STEP II, 2021, Q11 MS

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(i) $P_2 = \frac{1}{2}$ **B1**

T_3 can sit in seat S_3 if **M1**
T_1 chooses seat S_2, then T_2 chooses seat S_1
or T_1 chooses seat S_1
$P_3 = \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{2}$ **A1**

(ii) If passenger T_1 sits in seat S_k $(1 < k < n)$ then passengers T_2 to T_{k-1} all sit in their allocated seats. **E1**
The situation just before T_k arrives is then the same as for a train that did not have the $(k-1)$ seats that have been taken and for which T_k had been allocated seat S_1 **E1**

T_1 sits in seat S_1 with probability $\frac{1}{n}$, after which all the remaining passengers will get their allocated seats.

$$P(T_1 \text{ sits in } S_1 \cap T_n \text{ sits in } S_n) = \frac{1}{n}$$ **M1**

For $1 < k < n$, T_1 sits in seat S_k with probability $\frac{1}{n}$, so

$$P(T_1 \text{ sits in } S_k \cap T_n \text{ sits in } S_n) = \frac{1}{n} P_{n-k+1}$$ **M1**

If T_1 sits in S_n then it will not be possible for T_n to sit in S_n

$$P_n = \frac{1}{n} + \sum_{k=2}^{n-1} \frac{1}{n} P_{n-k+1} = \frac{1}{n}\left(1 + \sum_{r=2}^{n-1} P_r\right) \quad \textbf{AG}$$ **A1**

(iii) $P_n = \frac{1}{2}$ **B1**
Case where $n = 1$ is shown in part (i)
Suppose $P_k = \frac{1}{2}$ for $1 \leq k < n$:
$P_n = \frac{1}{n}\left(1 + (n-2) \times \frac{1}{2}\right) = \frac{1}{2}$ **M1** **A1**
Therefore, by induction $P_n = \frac{1}{2}$ **E1**

(iv) $Q_2 = \frac{1}{2}$ **B1**
For $n > 2$:
For $1 < k < n-1$:

$$P(T_{n-1} \text{ sits in } S_{n-1} | T_1 \text{ sits in } S_k) = Q_{n-k+1}$$ **M1**

(by similar reasoning as in part (ii))

$$P(T_{n-1} \text{ sits in } S_{n-1} | T_1 \text{ sits in } S_1 \text{ or } S_n) = 1$$ **M1**

Therefore

$$Q_n = \frac{1}{n}\left(2 + \sum_{k=2}^{n-2} Q_{n-k+1}\right) = \frac{1}{n}\left(2 + \sum_{r=3}^{n-1} Q_{n-k+1}\right)$$ **A1**

Base case:
If $n = 3$, then T_2 sits in seat S_2 in any case where T_1 does not sit in seat S_2 **B1**
Suppose $Q_k = \frac{2}{3}$ for some $3 \leq k < n$: **M1**
$Q_n = \frac{1}{n}\left(2 + (n-3) \times \frac{2}{3}\right) = \frac{2}{3}$ **A1**
Therefore, by induction $Q_n = \frac{2}{3}$ for $n \geq 3$ **E1**