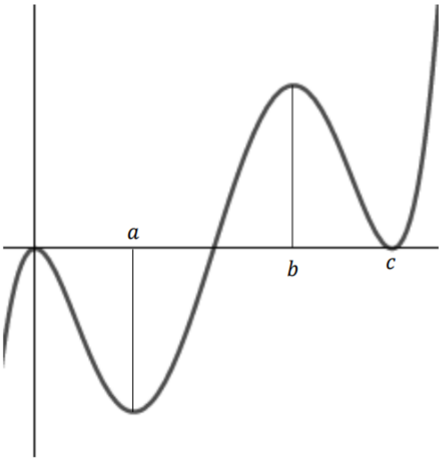
STEP II, 2020, Q8 MS

8(i)	
	Graph: Zeroes at $x = 0, c$ and one other point (h: label not required) in (a, b).
	Graph: Turning points at $x = 0, a, b, c$.
	Graph: Quintic shape with curve below axis in $(0, h)$ and above axis in (h, c)
	The area conditions give $F(0) = F(c) = 0$. $F'(x) = f(x)$, so $F'(0) = F'(a) = F'(b) = F'(c) = 0$
	Since f is a quartic and the coefficient of x^4 is 1, F must be a quintic and the coefficient of x^5 is $\frac{1}{5}$. $F(0) = F'(0) = 0$ and $F(c) = F'(c) = 0$, so F must have double roots at $x = 0$ and c. So $F(x)$ must have the given form. [Explanation must be clear that the double roots are deduced from the fact that $\mathrm{F(x) = F'(x) = 0}$ at those points.]
	$F(x) + F(c - x) = \frac{1}{5}x^2(x - c)^2[(x - h) + (c - x - h)]$ $= \frac{1}{5}x^2(c - x)^2(c - 2h)$
8(ii)	Let A be the (positive) area enclosed by the curve between 0 and a. The maximum turning point of $F(x)$ occurs at $x = b$, with $F(b) = A$. The minimum turning point of $F(x)$ occurs at $x = a$, with $F(a) = -A$.
	Therefore $F(x) \geq -A$, with equality iff $x = a$. So $F(b) + F(x) \geq 0$, with equality iff $x = a$.
	$F(a) + F(x) \leq 0$, with equality iff $x = b$.
	Since $F(b) + F(c - b) = \frac{1}{5}b^2(c - b)^2(c - 2h)$, either $c > 2h$, or $c = 2h$ and $c - b = a$.
	Also, $F(a) + F(c - a) = \frac{1}{5}a^2(c - a)^2(c - 2h)$, so either $c < 2h$, or $c = 2h$ and $c - a = b$.
	Thus $c = a + b$ and $c = 2h$.
8(iii)	$F(x) = \frac{1}{10}x^2(x - c)^2(2x - c)$ So $f(x) = \frac{1}{5}x(x - c)(5x^2 - 5xc + c^2)$

	The roots of the quadratic factor must be a and b.
	$f(x) = \frac{1}{5}(5x^4 - 10cx^3 + 6c^2x^2 - c^3x)$ $f'(x) = \frac{1}{5}(20x^3 - 30cx^2 + 12c^2)$ $f''(x) = \frac{1}{5}(60x^2 - 60cx + 12c^2) = \frac{12}{5}(5x^2 - 5cx + c^2)$
	Therefore $f''(x) = 0$ at $x = a$ and $x = b$ and so $(a, 0)$ and $(b, 0)$ are points of inflection.