STEP II, 2020, Q8

8 In this question, $\mathrm{f}(x)$ is a quartic polynomial where the coefficient of x^4 is equal to 1, and which has four real roots, 0, a, b and c, where $0 < a < b < c$.

$\mathrm{F}(x)$ is defined by $\mathrm{F}(x) = \displaystyle\int_0^x \mathrm{f}(t)\,\mathrm{d}t$.

The area enclosed by the curve $y = \mathrm{f}(x)$ and the x-axis between 0 and a is equal to that between b and c, and half that between a and b.

(i) Sketch the curve $y = \mathrm{F}(x)$, showing the x co-ordinates of its turning points.

Explain why $\mathrm{F}(x)$ must have the form $\mathrm{F}(x) = \frac{1}{5}x^2(x-c)^2(x-h)$, where $0 < h < c$.

Find, in factorised form, an expression for $\mathrm{F}(x) + \mathrm{F}(c-x)$ in terms of c, h and x.

(ii) If $0 \leqslant x \leqslant c$, explain why $\mathrm{F}(b) + \mathrm{F}(x) \geqslant 0$ and why $\mathrm{F}(b) + \mathrm{F}(x) > 0$ if $x \neq a$.
Hence show that $c - b = a$ or $c > 2h$.

By considering also $\mathrm{F}(a) + \mathrm{F}(x)$, show that $c = a + b$ and that $c = 2h$.

(iii) Find an expression for $\mathrm{f}(x)$ in terms of c and x only.

Show that the points of inflection on $y = \mathrm{f}(x)$ lie on the x-axis.