STEP II, 2020, Q6 MS

6(i)	Let $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then $\mathbf{M}^2 = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix}$
	so $\text{Tr}(\mathbf{M}^2) = a^2 + d^2 + 2bc = (a+d)^2 - 2(ad - bc)$
6(ii)	Let $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then $\mathbf{M}^2 = \begin{pmatrix} a\tau-\delta & b\tau \\ c\tau & d\tau-\delta \end{pmatrix}$, where $\tau = \text{Tr}(\mathbf{M})$ and $\delta = \text{Det}(\mathbf{M})$.
	Thus $\mathbf{M}^2 = \pm\mathbf{I} \Leftrightarrow \tau = 0$ and $\delta = \mp 1$ or $b = c = 0$ and $a^2 = d^2 = \pm 1$
	If $b = c = 0$ and $a = d = \pm 1$, then $\mathbf{M} = \pm\mathbf{I}$
	If $b = c = 0$ and $a = -d = \pm 1$, then $\tau = 0$ and $\delta = -1$
	Thus $\mathbf{M}^2 = +\mathbf{I} \Leftrightarrow \tau = 0$ and $\delta = -1$.
	Thus $\mathbf{M}^2 = -\mathbf{I} \Leftrightarrow \tau = 0$ and $\delta = +1$.
6(iii)	Part **(ii)** implies $\text{Det}(\mathbf{M}^2) = -1$, if $\mathbf{M}^4 = \mathbf{I}$, but $\mathbf{M}^2 \neq \pm\mathbf{I}$.
	However, $\text{Det}(\mathbf{M}^2) = \text{Det}(\mathbf{M})^2$, so this is impossible.
	Clearly $\mathbf{M}^2 = \pm\mathbf{I} \Rightarrow \mathbf{M}^4 = \mathbf{I}$
	Part **(ii)** implies that $\mathbf{M}^4 = -\mathbf{I} \Leftrightarrow \text{Tr}(\mathbf{M}^2) = 0$ and $\text{Det}(\mathbf{M}^2) = 1$
	so from **(i)** $\Leftrightarrow \text{Tr}(\mathbf{M})^2 = 2\text{Det}(\mathbf{M})$ and $\text{Det}(\mathbf{M}) = \pm 1$
	so $\Leftrightarrow \text{Tr}(\mathbf{M}) = \pm\sqrt{2}$ and $\text{Det}(\mathbf{M}) = 1$.
	Any example, for instance a matrix satisfying the conditions for any of $\mathbf{M}^2 = \mathbf{I}$, $\mathbf{M}^2 = -\mathbf{I}$, $\mathbf{M}^4 = -\mathbf{I}$, which is not a rotation or reflection.