

## STEP II, 2020, Q6

6 A  $2 \times 2$  matrix  $\mathbf{M}$  is real if it can be written as  $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , where  $a, b, c$  and  $d$  are real.

In this case, the *trace* of matrix  $\mathbf{M}$  is defined to be  $\text{tr}(\mathbf{M}) = a + d$  and  $\det(\mathbf{M})$  is the determinant of matrix  $\mathbf{M}$ . In this question,  $\mathbf{M}$  is a real  $2 \times 2$  matrix.

(i) Prove that

$$\text{tr}(\mathbf{M}^2) = \text{tr}(\mathbf{M})^2 - 2\det(\mathbf{M}).$$

(ii) Prove that

$$\mathbf{M}^2 = \mathbf{I} \text{ but } \mathbf{M} \neq \pm\mathbf{I} \iff \text{tr}(\mathbf{M}) = 0 \text{ and } \det(\mathbf{M}) = -1,$$

and that

$$\mathbf{M}^2 = -\mathbf{I} \iff \text{tr}(\mathbf{M}) = 0 \text{ and } \det(\mathbf{M}) = 1.$$

(iii) Use part (ii) to prove that

$$\mathbf{M}^4 = \mathbf{I} \iff \mathbf{M}^2 = \pm\mathbf{I}.$$

Find a necessary and sufficient condition on  $\det(\mathbf{M})$  and  $\text{tr}(\mathbf{M})$  so that  $\mathbf{M}^4 = -\mathbf{I}$ .

(iv) Give an example of a matrix  $\mathbf{M}$  for which  $\mathbf{M}^8 = \mathbf{I}$ , but which does not represent a rotation or reflection. [Note that the matrices  $\pm\mathbf{I}$  are both rotations.]



# NextStepMaths.com

To view mark schemes, fully worked solutions and examiner's comments, and for more details about tutoring and other services offered, go to [NextStepMaths.com](https://www.NextStepMaths.com)