STEP II, 2020, Q6

6 A 2×2 matrix $\mathbf{M}$ is real if it can be written as $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are real.

In this case, the *trace* of matrix $\mathbf{M}$ is defined to be $\text{tr}(\mathbf{M}) = a + d$ and $\det(\mathbf{M})$ is the determinant of matrix $\mathbf{M}$. In this question, $\mathbf{M}$ is a real 2×2 matrix.

(i) Prove that

$$\text{tr}(\mathbf{M}^2) = \text{tr}(\mathbf{M})^2 - 2\det(\mathbf{M}).$$

(ii) Prove that

$$\mathbf{M}^2 = \mathbf{I} \text{ but } \mathbf{M} \neq \pm\mathbf{I} \iff \text{tr}(\mathbf{M}) = 0 \text{ and } \det(\mathbf{M}) = -1,$$

and that

$$\mathbf{M}^2 = -\mathbf{I} \iff \text{tr}(\mathbf{M}) = 0 \text{ and } \det(\mathbf{M}) = 1.$$

(iii) Use part **(ii)** to prove that

$$\mathbf{M}^4 = \mathbf{I} \iff \mathbf{M}^2 = \pm\mathbf{I}.$$

Find a necessary and sufficient condition on $\det(\mathbf{M})$ and $\text{tr}(\mathbf{M})$ so that $\mathbf{M}^4 = -\mathbf{I}$.

(iv) Give an example of a matrix $\mathbf{M}$ for which $\mathbf{M}^8 = \mathbf{I}$, but which does not represent a rotation or reflection. [Note that the matrices $\pm\mathbf{I}$ are both rotations.]