# STEP II, 2020, Q5 MS

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| 5(i) | $x - q(x) = \sum_{r=0}^{n-1} a_r \times 10^r - \sum_{r=0}^{n-1} a_r = \sum_{r=0}^{n-1} a_r \times (10^r - 1)$ |
| | $10^r \geq 1 \quad \forall r$, so $x - q(x)$ is non-negative |
| | $9 \mid (10^r - 1) \quad \forall r$ |
| 5(ii) | $x - 44q(x) = 44(x - q(x)) = 43x$ |
| | So it is a multiple of 9 iff $43x$ is. |
| | $(43,9) = 1$, so $x - 44q(x)$ is a multiple of 9 iff $x$ is |
| | If $x$ has $n$ digits, $q(x) \leq 9n$ |
| | Since $x = 44q(x)$, $x \leq 396n$.<br>Any $n$ digit number must be at least $10^{n-1}$. |
| | These inequalities cannot be simultaneously true for $n \geq 5$ $(396 \times 5 < 10^4)$.<br>Therefore $n \leq 4$. |
| | Since $x - 44q(x) = 0$, which is a multiple of 9, $x$ is a multiple of 9. |
| | $q(x)$ is an integer and $x = 44q(x)$, so $x$ is a multiple of 44.<br>Since $(9,44) = 1$, $x$ must be a multiple of $44 \times 9 = 396$. |
| | So $x = 396k$ and therefore (by the result above) $k \leq 4$. |
| | Checking: Only $k = 2$ works. |
| 5(iii) | $x - 107q(q(x)) = 0 = 107(x - q(x)) + 107\left(q(x) - q(q(x))\right) - 106x$ |
| | $(x - q(x))$ and $(q(x) - q(q(x)))$ are both divisible by 9 (by part (i))<br>and so $x$ is divisible by 9 |
| | $x = 107q(q(x))$ and so is divisible by 107, and so is divisible by 963.<br>So $x = 963k$ for some $k$. |
| | If $x$ has $n$ digits, then $q(x) \leq 9n$. By (i), $q(q(x)) \leq q(x) \leq 9n$.<br>So $x \leq 963n$ and $x \geq 10^{n-1}$ which implies that $n \leq 4$ and so $k \leq 4$ |
| | Checking: Only $k = 1$ works. |