STEP II, 2020, Q4

4 **(i)** Given that a, b and c are the lengths of the sides of a triangle, explain why $c < a + b$, $a < b + c$ and $b < a + c$.

(ii) Use a diagram to show that the converse of the result in part **(i)** also holds: if a, b and c are positive numbers such that $c < a + b$, $a < b + c$ and $b < c + a$ then it is possible to construct a triangle with sides of length a, b and c.

(iii) When a, b and c are the lengths of the sides of a triangle, determine in each case whether the following sets of three lengths can

- always
- sometimes but not always
- never

form the sides of a triangle. Prove your claims.

(A) $a + 1,\ b + 1,\ c + 1.$

(B) $\dfrac{a}{b},\ \dfrac{b}{c},\ \dfrac{c}{a}.$

(C) $|a - b|,\ |b - c|,\ |c - a|.$

(D) $a^2 + bc,\ b^2 + ca,\ c^2 + ab.$

(iv) Let f be a function defined on the positive real numbers and such that, whenever $x > y > 0$,

$$f(x) > f(y) > 0 \text{ but } \frac{f(x)}{x} < \frac{f(y)}{y}.$$

Show that, whenever a, b and c are the lengths of the sides of a triangle, then $f(a)$, $f(b)$ and $f(c)$ can also be the lengths of the sides of a triangle.