STEP II, 2020, Q3 MS

3(i)	Suppose, $\exists k: 2 \le k \le n-1$ such that $u_{k-1} \ge u_k$, but $u_k < u_{k+1}$
	Since all of the terms are positive, these imply that $u_k^2 < u_{k-1}u_{k+1}$, so the sequence does not have property L.
	Therefore, if the sequence has property L, once a value k has been reached such that $u_{k-1} \ge u_k$, it must be the case that all subsequent terms also have that property (which is the given definition of unimodality).
3(ii)	$u_r - \alpha u_{r-1} = \alpha\left(u_{r-1} - \alpha u_{r-2}\right)$, so $u_r - \alpha u_{r-1} = \alpha^{r-2}\left(u_2 - \alpha u_1\right)$
	$u_r^2 - u_{r-1}u_{r+1} = u_r^2 - u_{r-1}\left(2\alpha u_r - \alpha^2 u_{r-1}\right) = \left(u_r - \alpha u_{r-1}\right)^2$ for $r \geqslant 2$
	The first identity shows that $u_r > 0$ for all r if $u_2 > au_1 > 0$.
	Since the right hand side of the second identity is always non-negative, the sequence has property L, and is hence unimodal.
3(iii)	$u_1 = (2-1)\alpha^{1-1} + 2(1-1)\alpha^{1-2} = 1$, which is correct. $u_2 = (2-2)\alpha^{2-1} + 2(2-1)\alpha^{2-2} = 2$, which is correct.
	Suppose that: $u_{k-2} = (4-k)\alpha^{k-3} + 2(k-3)\alpha^{k-4}$, and $u_{k-1} = (3-k)\alpha^{k-2} + 2(k-2)\alpha^{k-3}$.
	$u_k = 2\alpha\left((3-k)\alpha^{k-2} + 2(k-2)\alpha^{k-3}\right) - \alpha^2((4-k)\alpha^{k-3} + 2(k-3)\alpha^{k-4})$ $= \alpha^{k-1}(6-2k-4+k) + \alpha^{k-2}(4k-8-2k+6)$ $= \alpha^{k-1}(2-k) + 2\alpha^{k-2}(k-1)$ which is the correct expression for u_k
	Hence, by induction $u_r = (2-r)\alpha^{r-1} + 2(r-1)\alpha^{r-2}$
	$u_r - u_{r+1} = \left((2-r)\alpha^{r-1} + 2(r-1)\alpha^{r-2}\right) - \left((1-r)\alpha^r + 2r\alpha^{r-1}\right)$
	$= \alpha^{r-2}\left(2(r-1) + (2-3r)\alpha + (r-1)\alpha^2\right)$
	$= \dfrac{\alpha^{r-2}}{N^2}\left(2N^2(r-1) + (2-3r)N(N-1) + (r-1)(N-1)^2\right)$ $= \dfrac{\alpha^{r-2}}{N^2}\left((r-1) + rN - N^2\right)$
	when $r = N$, $u_N - u_{N+1} = \dfrac{\alpha^{r-2}(N-1)}{N^2} > 0$
	when $r = N-1$, $u_{N-1} - u_N = \dfrac{-2\alpha^{r-2}}{N^2} < 0$
	so u_r is largest when $r = N$