

STEP II, 2020, Q3

- 3 A sequence u_1, u_2, \dots, u_n of positive real numbers is said to be unimodal if there is a value k such that

$$u_1 \leq u_2 \leq \dots \leq u_k$$

and

$$u_k \geq u_{k+1} \geq \dots \geq u_n.$$

So the sequences 1, 2, 3, 2, 1; 1, 2, 3, 4, 5; 1, 1, 3, 3, 2 and 2, 2, 2, 2, 2 are all unimodal, but 1, 2, 1, 3, 1 is not.

A sequence u_1, u_2, \dots, u_n of positive real numbers is said to have property L if $u_{r-1}u_{r+1} \leq u_r^2$ for all r with $2 \leq r \leq n-1$.

- (i) Show that, in any sequence of positive real numbers with property L ,

$$u_{r-1} \geq u_r \implies u_r \geq u_{r+1}.$$

Prove that any sequence of positive real numbers with property L is unimodal.

- (ii) A sequence u_1, u_2, \dots, u_n of real numbers satisfies $u_r = 2\alpha u_{r-1} - \alpha^2 u_{r-2}$ for $3 \leq r \leq n$, where α is a positive real constant. Prove that, for $2 \leq r \leq n$,

$$u_r - \alpha u_{r-1} = \alpha^{r-2}(u_2 - \alpha u_1)$$

and, for $2 \leq r \leq n-1$,

$$u_r^2 - u_{r-1}u_{r+1} = (u_r - \alpha u_{r-1})^2.$$

Hence show that the sequence consists of positive terms and is unimodal, provided $u_2 > \alpha u_1 > 0$.

In the case $u_1 = 1$ and $u_2 = 2$, prove by induction that $u_r = (2-r)\alpha^{r-1} + 2(r-1)\alpha^{r-2}$.

Let $\alpha = 1 - \frac{1}{N}$, where N is an integer with $2 \leq N \leq n$.

In the case $u_1 = 1$ and $u_2 = 2$, prove that u_r is largest when $r = N$.



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