STEP II, 2020, Q3

3 A sequence $u_1, u_2, \ldots, u_n$ of positive real numbers is said to be unimodal if there is a value k such that

$$u_1 \leqslant u_2 \leqslant \ldots \leqslant u_k$$

and

$$u_k \geqslant u_{k+1} \geqslant \ldots \geqslant u_n.$$

So the sequences $1, 2, 3, 2, 1$; $1, 2, 3, 4, 5$; $1, 1, 3, 3, 2$ and $2, 2, 2, 2, 2$ are all unimodal, but $1, 2, 1, 3, 1$ is not.

A sequence $u_1, u_2, \ldots, u_n$ of positive real numbers is said to have property L if $u_{r-1}u_{r+1} \leqslant u_r^2$ for all r with $2 \leqslant r \leqslant n-1$.

(i) Show that, in any sequence of positive real numbers with property L,

$$u_{r-1} \geqslant u_r \implies u_r \geqslant u_{r+1}.$$

Prove that any sequence of positive real numbers with property L is unimodal.

(ii) A sequence $u_1, u_2, \ldots, u_n$ of real numbers satisfies $u_r = 2\alpha u_{r-1} - \alpha^2 u_{r-2}$ for $3 \leqslant r \leqslant n$, where α is a positive real constant. Prove that, for $2 \leqslant r \leqslant n$,

$$u_r - \alpha u_{r-1} = \alpha^{r-2}(u_2 - \alpha u_1)$$

and, for $2 \leqslant r \leqslant n-1$,

$$u_r^2 - u_{r-1}u_{r+1} = (u_r - \alpha u_{r-1})^2.$$

Hence show that the sequence consists of positive terms and is unimodal, provided $u_2 > \alpha u_1 > 0$.

In the case $u_1 = 1$ and $u_2 = 2$, prove by induction that $u_r = (2-r)\alpha^{r-1} + 2(r-1)\alpha^{r-2}$.

Let $\alpha = 1 - \dfrac{1}{N}$, where N is an integer with $2 \leqslant N \leqslant n$.

In the case $u_1 = 1$ and $u_2 = 2$, prove that u_r is largest when $r = N$.