STEP II, 2020, Q2

2 The curves C_1 and C_2 both satisfy the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{kxy - y}{x - kxy},$$

where $k = \ln 2$.

All points on C_1 have positive x and y co-ordinates and C_1 passes through $(1\,,\ 1)$. All points on C_2 have negative x and y co-ordinates and C_2 passes through $(-1\,,\ -1)$.

(i) Show that the equation of C_1 can be written as $(x-y)^2 = (x+y)^2 - 2^{x+y}$.

Determine a similar result for curve C_2.

Hence show that $y = x$ is a line of symmetry of each curve.

(ii) Sketch on the same axes the curves $y = x^2$ and $y = 2^x$, for $x \geqslant 0$. Hence show that C_1 lies between the lines $x + y = 2$ and $x + y = 4$.

Sketch curve C_1.

(iii) Sketch curve C_2.