

STEP II, 2020, Q12 MS

12(i)	For the biased die: $P(R_1 = R_2) = \sum_{i=1}^n \left(\frac{1}{n} + \varepsilon_i\right)^2$
	$P(R_1 = R_2) = \frac{1}{n^2} \sum_{i=1}^n 1 + \frac{2}{n} \sum_{i=1}^n \varepsilon_i + \sum_{i=1}^n \varepsilon_i^2$
	$\sum_{i=1}^n \varepsilon_i = 0$, so $P(R_1 = R_2) = \frac{1}{n} + \sum_{i=1}^n \varepsilon_i^2$
	For a fair die, $P(R_1 = R_2) = \frac{1}{n}$ and $\sum_{i=1}^n \varepsilon_i^2 > 0$, so it is more likely with the biased die.
(ii)	$P(R_1 > R_2) = \frac{1}{2}(1 - P(R_1 = R_2))$
	Therefore, the value of $P(R_1 > R_2)$ if the die is possibly biased is $\leq P(R_1 > R_2)$ if the die is fair.
	Let $T = \sum_{r=1}^n x_r$ and, for each i , let $p_i = \frac{x_i}{T}$ Then $\sum_{i=1}^n p_i = 1$, so we can construct a biased n -sided die with $P(X = i) = p_i$
	$P(R_1 > R_2) = \sum_{i=2}^n \sum_{j=1}^{i-1} p_i p_j$
	For a fair die: $P(R_1 > R_2) = \frac{n-1}{2n}$
	Therefore $\sum_{i=2}^n \sum_{j=1}^{i-1} \frac{x_i x_j}{T^2} \leq \frac{n-1}{2n}$ and so $\sum_{i=2}^n \sum_{j=1}^{i-1} x_i x_j \leq \frac{n-1}{2n} \left(\sum_{i=1}^n x_i \right)^2$
(iii)	For the biased die: $P(R_1 = R_2 = R_3) = \sum_{i=1}^n \left(\frac{1}{n} + \varepsilon_i\right)^3$
	$= \sum_{i=1}^n \frac{1}{n^3} + \sum_{i=1}^n \frac{3\varepsilon_i}{n^2} + \sum_{i=1}^n \frac{3\varepsilon_i^2}{n} + \sum_{i=1}^n \varepsilon_i^3$
	Therefore $P(R_1 = R_2 = R_3 \text{ biased}) - P(R_1 = R_2 = R_3 \text{ fair}) = \sum_{i=1}^n \frac{3\varepsilon_i}{n^2} + \sum_{i=1}^n \frac{3\varepsilon_i^2}{n} + \sum_{i=1}^n \varepsilon_i^3$ $= \sum_{i=1}^n \frac{3\varepsilon_i^2}{n} + \sum_{i=1}^n \varepsilon_i^3$ (since $\sum_{i=1}^n \varepsilon_i = 0$)
	$= \sum_{i=1}^n \frac{3\varepsilon_i^2}{n} + \varepsilon_i^3 = \sum_{i=1}^n \varepsilon_i^2 \left(\frac{3}{n} + \varepsilon_i\right)$
	But $\varepsilon_i \geq -\frac{1}{n}$ (since $p_i \geq 0$), so this sum must be positive.
	Therefore, $P(R_1 = R_2 = R_3)$ must be greater for the biased die.



NextStepMaths.com

To view mark schemes, fully worked solutions and examiner's comments, and for more details about tutoring and other services offered, go to [NextStepMaths.com](https://www.NextStepMaths.com)