STEP II, 2020, Q12 MS

12(i)	For the biased die: $P(R_1 = R_2) = \sum_{i=1}^{n}\left(\frac{1}{n} + \varepsilon_i\right)^2$
	$P(R_1 = R_2) = \frac{1}{n^2}\sum_{i=1}^{n} 1 + \frac{2}{n}\sum_{i=1}^{n}\varepsilon_i + \sum_{i=1}^{n}\varepsilon_i{}^2$
	$\sum_{i=1}^{n}\varepsilon_i = 0$, so $P(R_1 = R_2) = \frac{1}{n} + \sum_{i=1}^{n}\varepsilon_i{}^2$
	For a fair die, $P(R_1 = R_2) = \frac{1}{n}$ and $\sum_{i=1}^{n}\varepsilon_i{}^2 > 0$, so it is more likely with the biased die.
(ii)	$P(R_1 > R_2) = \frac{1}{2}\left(1 - P(R_1 = R_2)\right)$
	Therefore, the value of $P(R_1 > R_2)$ if the die is possibly biased is $\leq P(R_1 > R_2)$ if the die is fair.
	Let T= $\sum_{r=1}^{n} x_r$ and, for each i, let $p_i = \frac{x_i}{T}$ Then $\sum_{i=1}^{n} p_i = 1$, so we can construct a biased n-sided die with $P(X = i) = p_i$
	$P(R_1 > R_2) = \sum_{i=2}^{n}\sum_{j=1}^{i-1} p_i p_j$
	For a fair die: $P(R_1 > R_2) = \frac{n-1}{2n}$
	Therefore $\sum_{i=2}^{n}\sum_{j=1}^{i-1}\frac{x_i x_j}{T^2} \leq \frac{n-1}{2n}$ and so $$\sum_{i=2}^{n}\sum_{j=1}^{i-1} x_i x_j \leq \frac{n-1}{2n}\left(\sum_{i=1}^{n} x_i\right)^2$$
(iii)	For the biased die: $P(R_1 = R_2 = R_3) = \sum_{i=1}^{n}\left(\frac{1}{n} + \varepsilon_i\right)^3$
	$= \sum_{i=1}^{n}\frac{1}{n^3} + \sum_{i=1}^{n}\frac{3\varepsilon_i}{n^2} + \sum_{i=1}^{n}\frac{3\varepsilon_i^2}{n} + \sum_{i=1}^{n}\varepsilon_i{}^3$
	Therefore $$P(R_1 = R_2 = R_3\ biased) - P(R_1 = R_2 = R_3\ fair) = \sum_{i=1}^{n}\frac{3\varepsilon_i}{n^2} + \sum_{i=1}^{n}\frac{3\varepsilon_i^2}{n} + \sum_{i=1}^{n}\varepsilon_i{}^3$$ $= \sum_{i=1}^{n}\frac{3\varepsilon_i^2}{n} + \sum_{i=1}^{n}\varepsilon_i{}^3$ (since $\sum_{i=1}^{n}\varepsilon_i = 0$)
	$= \sum_{i=1}^{n}\frac{3\varepsilon_i^2}{n} + \varepsilon_i^3 = \sum_{i=1}^{n}\varepsilon_i^2\left(\frac{3}{n} + \varepsilon_i\right)$
	But $\varepsilon_i \geq -\frac{1}{n}$ (since $p_i \geq 0$), so this sum must be positive.
	Therefore, $P(R_1 = R_2 = R_3)$ must be greater for the biased die.