# STEP II, 2020, Q11 MS

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| 11(i) | If the game has not ended after $2n$ turns, then the sequence has either been $n$ repetitions of $HT$ or $n$ repetitions of $TH$.<br>So $P(Game\ has\ not\ finished\ after\ 2n\ turns) = 2(pq)^n$.<br>So the probability that the game never ends is $\lim_{n\to\infty} 2(pq)^n = 0$. |
| | Sequence that follows the first $H$ will be $k$ repetitions of $TH$, followed by $H$, where $k \geq 0$. |
| | So $P(A\ wins \mid first\ toss\ is\ H) = \sum_{k=0}^{\infty}(pq)^k p = \frac{p}{1-pq}$ |
| | $P(A\ wins \cap first\ toss\ is\ H) = p \times \frac{p}{1-pq}$ |
| | If first toss is a tail then the sequence that follows would be<br>$k$ repetitions of $HT$ followed by $HH$. |
| | So $P(A\ wins \mid first\ toss\ is\ T) = \frac{p^2}{1-pq}$ |
| | $P(A\ wins \cap first\ toss\ is\ T) = \frac{p^2 q}{1-pq}$ |
| | Therefore $P(A\ wins) = \frac{p^2(1+q)}{1-pq}$ |
| 11(ii) | Following a first toss of $H$:<br>$A$ wins with $HH$<br>or ($HT$ followed by any sequence where A wins after first toss was $T$)<br>or ($T$ followed by any sequence where A wins after first toss was $T$) |
| | The probabilities of these cases are:<br>$p^2$<br>$pq\ P(A$ wins \|the first toss is a tail)<br>$q\ P(A$ wins \|the first toss is a tail) |
| | Therefore:<br>$P(\text{A wins} \mid \text{the first toss is a head}) = p^2 + (q + pq)P(\text{A wins} \mid \text{the first toss is a tail})$ |

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| | Similarly, following first toss of $T$:<br>$A$ wins with ($H$ followed by any sequence where A wins after first toss was H) or ($TH$ followed by any sequence where A wins after first toss was $H$) |
| | Therefore:<br>$P(\text{A wins} \mid \text{the first toss is a tail}) = (p + pq)P(\text{A wins} \mid \text{the first toss is a head})$ |
| | So<br>$P(A \mid H\ first) = p^2 + (q + pq)(p + pq)P(A \mid H\ first)$<br>$P(A \mid H\ first) = \frac{p^2}{1-(p+pq)(q+pq)}$ |
| | And<br>$P(A \mid T\ first) = (p + pq)(p^2 + (q + pq)P(A \mid T\ first))$<br>$P(A \mid T\ first) = \frac{p^2(p+pq)}{1-(p+pq)(q+pq)}$ |
| | So<br>$P(A\ wins) = p \times \frac{p^2}{1-(p+pq)(q+pq)} + q \times \frac{p^2(p+pq)}{1-(p+pq)(q+pq)} = \frac{p^2(1-q^3)}{1-(1-p^2)(1-q^2)}$ |
| 11(iii) | Let $W$ be the event that $A$ wins the game.<br>$P(W \mid H\ first) = p^{a-1} + (1 + p + p^2 + \cdots + p^{a-2})qP(W \mid T\ first)$ |
| | $P(W \mid T\ first) = (1 + q + q^2 + \cdots + q^{b-2})pP(W \mid H\ first)$ |
| | $P(W \mid H\ first) = \frac{p^{a-1}}{1-(1-p^{a-1})(1-q^{b-1})}$ |
| | $P(W \mid T\ first) = \frac{p^{a-1}(1-q^{b-1})}{1-(1-p^{a-1})(1-q^{b-1})}$ |
| | Therefore:<br>$P(W) = \frac{p^{a-1}(1-q^b)}{1-(1-p^{a-1})(1-q^{b-1})}$ |
| | If $a = b = 2$,<br>$P(W) = \frac{p(1-q^2)}{1-(1-p)(1-q)} = \frac{p^2(1+q)}{1-pq}$ as expected. |