STEP II, 2020, Q10

10 A particle P of mass m moves freely and without friction on a wire circle of radius a, whose axis is horizontal. The highest point of the circle is H, the lowest point of the circle is L and angle $PHL = \theta$. A light spring of modulus of elasticity λ is attached to P and to H. The natural length of the spring is l, which is less than the diameter of the circle.

(i) Show that, if there is an equilibrium position of the particle at $\theta = \alpha$, where $\alpha > 0$, then $\cos\alpha = \dfrac{\lambda l}{2(a\lambda - mgl)}$.

Show also that there will only be such an equilibrium position if $\lambda > \dfrac{2mgl}{2a - l}$.

When the particle is at the lowest point L of the circular wire, it has speed u.

(ii) Show that, if the particle comes to rest before reaching H, it does so when $\theta = \beta$, where $\cos\beta$ satisfies

$$(\cos\alpha - \cos\beta)^2 = (1 - \cos\alpha)^2 + \frac{mu^2}{2a\lambda}\cos\alpha,$$

where $\cos\alpha = \dfrac{\lambda l}{2(a\lambda - mgl)}$.

Show also that this will only occur if $u^2 < \dfrac{2a\lambda}{m}(2 - \sec\alpha)$.