STEP II, 2019, Q8

8 The domain of the function f is the set of all 2×2 matrices and its range is the set of real numbers. Thus, if $\mathbf{M}$ is a 2×2 matrix, then $\mathrm{f}(\mathbf{M}) \in \mathbb{R}$.

The function f has the property that $\mathrm{f}(\mathbf{MN}) = \mathrm{f}(\mathbf{M})\mathrm{f}(\mathbf{N})$ for any 2×2 matrices $\mathbf{M}$ and $\mathbf{N}$.

(i) You are given that there is a matrix $\mathbf{M}$ such that $\mathrm{f}(\mathbf{M}) \neq 0$. Let $\mathbf{I}$ be the 2×2 identity matrix. By considering $\mathrm{f}(\mathbf{MI})$, show that $\mathrm{f}(\mathbf{I}) = 1$.

(ii) Let $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. You are given that $\mathrm{f}(\mathbf{J}) \neq 1$. By considering $\mathbf{J}^2$, evaluate $\mathrm{f}(\mathbf{J})$.

Using $\mathbf{J}$, show that, for any real numbers a, b, c and d,

$$\mathrm{f}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = -\mathrm{f}\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right) = \mathrm{f}\left(\begin{pmatrix} d & c \\ b & a \end{pmatrix}\right).$$

(iii) Let $\mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ where $k \in \mathbb{R}$. Use $\mathbf{K}$ to show that, if the second row of the matrix $\mathbf{A}$ is a multiple of the first row, then $\mathrm{f}(\mathbf{A}) = 0$.

(iv) Let $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. By considering the matrices $\mathbf{P}^2$, $\mathbf{P}^{-1}$, and $\mathbf{K}^{-1}\mathbf{PK}$ for suitable values of k, evaluate $\mathrm{f}(\mathbf{P})$.