STEP II, 2019, Q7 MS

In part (i), taking the scalar product of $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ with each of the vectors in turn produces a set of three equations from which it can be deduced that $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{c} \cdot \boldsymbol{a}$ and that any pair of them add up to -1. Alternatively, it can be observed that $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) = (-\boldsymbol{c}) \cdot (-\boldsymbol{c})$.

It can then be shown that the angle between any pair of these vectors is $120°$ and so a sketch shows that the triangle must be equilateral.

In part (ii), a similar approach will lead to the given result. Alternatively, the result can be obtained by observing that $(\boldsymbol{a}_1 + \boldsymbol{a}_2) \cdot (\boldsymbol{a}_1 + \boldsymbol{a}_2) = (-\boldsymbol{a}_3 - \boldsymbol{a}_4) \cdot (-\boldsymbol{a}_3 - \boldsymbol{a}_4)$. For part (a), note that it must be the case that the angle between any pair of vectors is equal to the angle between the other two vectors.

For part (b) use the vector $(\boldsymbol{a}_1 - \boldsymbol{a}_2)$ to find the length of one side of the tetrahedron. From the fact that the tetrahedron is regular it can be deduced that $\boldsymbol{a}_1 \cdot \boldsymbol{a}_2 = \boldsymbol{a}_1 \cdot \boldsymbol{a}_3 = \boldsymbol{a}_1 \cdot \boldsymbol{a}_4$. The side length can then be calculated.

7	(i)		$\boldsymbol{a}\cdot(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})=0$ $\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}=-1$ and cyclic permutations $\boldsymbol{a}\cdot\boldsymbol{b}=-\frac{1}{2}$ legitimately obtained $\cos\theta=-\frac{1}{2}$ where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ $\theta=120°$ Similarly, the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $120°$. Justification of equilateral triangle by sketch or otherwise ABC is equilateral	M1 M1 A1 M1 A1 M1 M1 A1 (8 marks)
	(ii)		$\boldsymbol{a_1}\cdot\boldsymbol{a_2}+\boldsymbol{a_1}\cdot\boldsymbol{a_3}+\boldsymbol{a_1}\cdot\boldsymbol{a_4}=-1$ and cyclic permutations Linear combination of these equations $\boldsymbol{a_1}\cdot\boldsymbol{a_2}=\boldsymbol{a_3}\cdot\boldsymbol{a_4}$ (legitimately obtained)	M1 M1 A1 (AG) (3 marks)
		(a)	Angles $\angle A_1OA_2=\angle A_3OA_4$ By symmetry, $\angle A_2OA_3=\angle A_4OA_1$ The $\boldsymbol{a_i}$ are distinct and unit length so no angles are zero (accept justification by sketch) $A_1A_2A_3A_4$ is a rectangle	M1 M1 A1 (3 marks)
		(b)	$(A_1A_2)^2=(\boldsymbol{a_1}-\boldsymbol{a_2})^2$ $=\boldsymbol{a_1^2}+\boldsymbol{a_1^2}-2\boldsymbol{a_1}\cdot\boldsymbol{a_2}$ $=2-2\boldsymbol{a_1}\cdot\boldsymbol{a_2}$ By symmetry, $\boldsymbol{a_1}\cdot\boldsymbol{a_2}=\boldsymbol{a_1}\cdot\boldsymbol{a_3}=\boldsymbol{a_1}\cdot\boldsymbol{a_4}$ So $\boldsymbol{a_1}\cdot\boldsymbol{a_2}=-\frac{1}{3}$ So $(A_1A_2)^2=\frac{8}{3}$ $A_1A_2=\frac{2\sqrt{2}}{\sqrt{3}}$	 M1 M1 M1 A1 M1 A1 (6 marks)