STEP II, 2019, Q5 MS

In part (i) the values of a for which the sequence is constant can be found by solving the equation $a = f(a)$. The sequence will have period 2 if the equation $a = f(f(a))$ has a solution that is different from those for which the sequence is constant. Although the equation $a = f(f(a))$ is a quartic, it is clear that the values of a for which the sequence is constant will be solutions of this equation as well. This means that two factors of the quartic are known and so the remaining factor will be a quadratic. When considering the roots of this quadratic it must also be checked to confirm that the roots are not repeats of the values that give a constant sequence.

In part (ii), note that there cannot be a solution to the equation $f(a) = a$ and so it must be the case that either $f(x) > x$ for all x or $f(x) < x$ for all x (since f is a continuous function). It is clear that $f(x) > x$ for large values of x.

Since it must be that case that $f(x) > x$ for all x if the sequence is not constant, it must also be the case that $f\big(f(x)\big) > x$ for all x.

Finally, it can be seen that, in the case where $q = p$ the sequence is of the form in part (i) and so it should be possible to deduce a case in which there is no value of a for which the sequence has period 2, but there is a value of a for which the sequence is constant.

5	(i)		Constant iff $a = f(a)$	M1
			$\Leftrightarrow a = p + (a - p)a$	
			$\Leftrightarrow 0 = (a - p)(a - 1)$	M1
			$\Leftrightarrow a = p$ or $a = 1$.	A1
			Period 2	
			$\Leftrightarrow a = f(f(a))$	M1
			$\Leftrightarrow 0 = (a - p)(-1 + 2ap - pa^2 + a^3)$ (factorisation)	M1
			$\Leftrightarrow 0 = (a - p)(a - 1)(a^2 + (1 - p)a + 1)$	A1
			If $a = p$ or $a = 1$, then sequence is constant.	B1
			The quadratic has solutions when $(p - 1)^2 \geq 4$.	M1
			If $(p - 1)^2 > 4$, i.e. $p > 3$ or $p < -1$, the solutions are distinct.	
			They are not both $1, p$ since the sum of the roots is $p - 1 \neq p + 1$	E1
			So for $p > 3$ or $p < -1$, one of the roots of the quadratic gives a sequence of period 2.	E1 (AG)
			If $p = 3$, $a = 1$ so not period 2.	B1
			If $p = -1$, $a = -1 = p$ so not period 2.	B1
				(12 marks)
	(ii)		No value of a for which the sequence is constant	
			$\Leftrightarrow f(a) = a$ has no solution	E1 (→)
			$\Leftrightarrow f(x) > x$ or $f(x) < x$ for all x	E1 (←)
			But $f(x) > x$ for large x.	
			So cannot have $f(x) < x$ for all x.	E1
			If no value of a for which sequence constant,	
			then $f(x) > x$ for all x	E1
			So $f\big(f(x)\big) > f(x) > x$ for all x	E1
			And hence no solution to $f\big(f(a)\big) = a$.	E1
			Setting $p = q$, gives (i).	E1
			Then if $-1 \leq p \leq 3$, there is no period 2 sequence but a constant sequence exists.	E1
				(8 marks)