STEP II, 2019, Q5

5 The sequence $u_0, u_1, \ldots$ is said to be a *constant sequence* if $u_n = u_{n+1}$ for $n = 0, 1, 2, \ldots$. The sequence is said to be a *sequence of period* 2 if $u_n = u_{n+2}$ for $n = 0, 1, 2, \ldots$ and the sequence is not constant.

(i) A sequence of real numbers is defined by $u_0 = a$ and $u_{n+1} = \mathrm{f}(u_n)$ for $n = 0, 1, 2, \ldots$, where

$$\mathrm{f}(x) = p + (x - p)x,$$

and p is a given real number.

Find the values of a for which the sequence is constant.

Show that the sequence has period 2 for some value of a if and only if $p > 3$ or $p < -1$.

(ii) A sequence of real numbers is defined by $u_0 = a$ and $u_{n+1} = \mathrm{f}(u_n)$ for $n = 0, 1, 2, \ldots$, where

$$\mathrm{f}(x) = q + (x - p)x,$$

and p and q are given real numbers.

Show that there is no value of a for which the sequence is constant if and only if $\mathrm{f}(x) > x$ for all x.

Deduce that, if there is no value of a for which the sequence is constant, then there is no value of a for which the sequence has period 2.

Is it true that, if there is no value of a for which the sequence has period 2, then there is no value of a for which the sequence is constant?