STEP II, 2019, Q4 MS

In the first part, if the expression to be evaluated is multiplied by $\sin\left(\frac{\pi}{9}\right)$, then three applications of the given identity can be used. A similar process can then be used to simplify the first expression in part (ii). For the sum, note that $\tan x$ is the derivative of $-\ln(\cos x)$ and so, the result can be obtained by taking logs of the first result and then differentiating term by term.

For the final part, first change to a finite product (from $k = 1$ to $k = n$) and then take the limit as $n \to \infty$. Note however, that the product in part (ii) started at $k = 0$, so the result in part (ii) needs to be modified before it can be applied.

In the same way, the sum can be modified to start at $k = 0$ (where $k = j - 2$) and then the result of part (ii) can be applied with $x = \frac{\pi}{4}$.

4	(i)		$\sin\frac{\pi}{9}\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$ $=\frac{1}{2}\sin\frac{2\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$ $=\frac{1}{8}\sin\frac{8\pi}{9}$ $=\frac{1}{8}\sin\frac{\pi}{9}$ (use of $\sin(\pi-x)=\sin(x)$) $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}=\frac{1}{8}$	B1 M1 M1 A1 (4 marks)
	(ii)		$\sin\left(\frac{x}{2^n}\right)\prod_{k=0}^{n}\cos\left(\frac{x}{2^k}\right)$ $=\frac{1}{2}\sin\left(\frac{x}{2^{n-1}}\right)\prod_{k=0}^{n-1}\cos\left(\frac{x}{2^k}\right)$ $=\cdots$ (convincing use of induction or repeated application) $=\frac{\sin(2x)}{2^{n+1}}$ (induction end point correct) $\prod_{k=0}^{n}\cos\left(\frac{x}{2^k}\right)=\frac{\sin(2x)}{2^{n+1}\sin\left(\frac{x}{2^n}\right)}$ $\sum_{k=0}^{n}\log\left(cos\left(\frac{x}{2^k}\right)\right)=\log(\sin(2x))-\log\left(sin\left(\frac{x}{2^n}\right)\right)-\log(2^{n+1})$ $\sum_{k=0}^{n}\frac{1}{2^k}\tan\left(\frac{x}{2^k}\right)=-2\cot(2x)+\frac{1}{2^n}\cot\left(\frac{x}{2^n}\right)$ (justified with differentiation)	B1 M1 E1 A1 M1 (diff) M1 (division) A1 (AG) (7 marks)
	(iii)		B1 – switch to product starting at 0 M1 – set up as limiting case of product to n M1 – apply small angle for sin A1 – correct answer $\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)=\frac{\sin(2x)}{2^{n+1}\sin\left(\frac{x}{2^n}\right)\cos(x)}$ $=\frac{2\sin(x)}{2^{n+1}\sin\left(\frac{x}{2^n}\right)}$ $\sim\frac{\sin(x)}{2^n\times\left(\frac{x}{2^n}\right)}$ $=\frac{\sin(x)}{x}$	M1 M1 M1 A1 (AG)

			$\sum_{j=2}^{n} \frac{1}{2^{j-2}} \tan\left(\frac{\pi}{2^j}\right)$ $= \sum_{k=0}^{n} \frac{1}{2^k} \tan\left(\frac{\pi/4}{2^k}\right)$ $= \lim_{n\to\infty}\left(\frac{1}{2^n}\cot\left(\frac{\pi/4}{2^n}\right) - 2\cot\left(\frac{\pi}{2}\right)\right)$ $= \lim_{n\to\infty}\left(\frac{1}{2^n\tan\left(\frac{\pi/4}{2^n}\right)}\right)$ $= \lim_{n\to\infty}\left(\frac{1}{2^n\left(\frac{\pi/4}{2^n}\right)}\right)$ $= \frac{4}{\pi}$	M1 M1 M1 M1 A1 (9 marks)

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