STEP II, 2019, Q4

4 You are not required to consider issues of convergence in this question.

For any sequence of numbers $a_1,\ a_2,\ \dots\ ,\ a_m,\ \dots\ ,\ a_n$, the notation $\displaystyle\prod_{i=m}^{n} a_i$ denotes the product $a_m a_{m+1}\cdots a_n$.

(i) Use the identity $2\cos x \sin x = \sin(2x)$ to evaluate the product $\cos(\frac{\pi}{9})\cos(\frac{2\pi}{9})\cos(\frac{4\pi}{9})$.

(ii) Simplify the expression

$$\prod_{k=0}^{n} \cos\left(\frac{x}{2^k}\right) \qquad (0 < x < \tfrac{1}{2}\pi).$$

Using differentiation, or otherwise, show that, for $0 < x < \frac{1}{2}\pi$,

$$\sum_{k=0}^{n} \frac{1}{2^k}\tan\left(\frac{x}{2^k}\right) = \frac{1}{2^n}\cot\left(\frac{x}{2^n}\right) - 2\cot(2x).$$

(iii) Using the results $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ and $\displaystyle\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1$, show that

$$\prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right) = \frac{\sin x}{x}$$

and evaluate

$$\sum_{j=2}^{\infty} \frac{1}{2^{j-2}}\tan\left(\frac{\pi}{2^j}\right).$$