

STEP II, 2019, Q4

4 You are not required to consider issues of convergence in this question.

For any sequence of numbers $a_1, a_2, \dots, a_m, \dots, a_n$, the notation $\prod_{i=m}^n a_i$ denotes the product $a_m a_{m+1} \cdots a_n$.

(i) Use the identity $2 \cos x \sin x = \sin(2x)$ to evaluate the product $\cos\left(\frac{\pi}{9}\right) \cos\left(\frac{2\pi}{9}\right) \cos\left(\frac{4\pi}{9}\right)$.

(ii) Simplify the expression

$$\prod_{k=0}^n \cos\left(\frac{x}{2^k}\right) \quad (0 < x < \frac{1}{2}\pi).$$

Using differentiation, or otherwise, show that, for $0 < x < \frac{1}{2}\pi$,

$$\sum_{k=0}^n \frac{1}{2^k} \tan\left(\frac{x}{2^k}\right) = \frac{1}{2^n} \cot\left(\frac{x}{2^n}\right) - 2 \cot(2x).$$

(iii) Using the results $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ and $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1$, show that

$$\prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right) = \frac{\sin x}{x}$$

and evaluate

$$\sum_{j=2}^{\infty} \frac{1}{2^{j-2}} \tan\left(\frac{\pi}{2^j}\right).$$



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