STEP II, 2019, Q3 MS

The initial result can be shown by considering the four possible combinations of signs for x_1 and x_2. Induction can then be used to prove the more general result.

In part (i)(a) the initial result can then be applied to show that the value of $f(x) - 1$ must be less than or equal to a polynomial in $|x|$. Furthermore, the coefficients must also be less than or equal to A and so the value must be less than or equal to a sum that can be seen to be a geometric sequence.

In part (i)(b) the previous result can be applied in the case $x = \omega$ and, since $|\omega| < 1$ it must be the case that $1 - |\omega| > 0$. Therefore, the inequality can be multiplied by $(1 - |\omega|)$ without changing the direction of the inequality. The required inequality then follows easily.

To show that the inequalities continue to hold if $|\omega| > 1$, observe that $\frac{1}{\omega}$ is a root of the polynomial $g(x) = 1 + a_{n-1}x + \cdots + a_1x^{n-1} + x^n$, as $g\left(\frac{1}{\omega}\right) = \frac{1}{\omega^n}f(\omega) = 0$. Since $g(x)$ has the same properties as $f(x)$, $\frac{1}{|\omega|}$ must also satisfy (*). It then only remains to consider the case $|\omega| = 1$.

For the final part, observe that division by 135 produces a polynomial that satisfies the conditions specified and so the bounds on the value of ω reduces the cases to be considered to $\omega = \pm 1$ and $\omega = \pm 2$.

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3			$\lvert x_1 + x_2\rvert$ is maximised when both have the same sign, In which case $\lvert x_1 + x_2\rvert = \lvert x_1\rvert + \lvert x_2\rvert$. Thus, $\lvert x_1 + x_2\rvert \le \lvert x_1 + x_2\rvert$ (or by consideration of all four combinations of signs separately) $\lvert x_1 + \dots + x_{n-1} + x_n\rvert \le \lvert x_1 + \dots + x_{n-1}\rvert + \lvert x_n\rvert$ $\le \cdots$ $\le \lvert x_1\rvert + \dots + \lvert x_{n-1}\rvert + \lvert x_n\rvert$ by induction	E1 E1 (2 marks)
	(i)	(a)	$\lvert f(x) - 1\rvert = \lvert a_1 x + \dots + a_{n-1}x^{n-1} + x^n\rvert$ $\le \lvert a_1 x\rvert + \dots + \lvert a_{n-1}x^{n-1}\rvert + \lvert x^n\rvert$ $= \lvert a_1\rvert\lvert x\rvert + \dots + \lvert a_{n-1}\rvert\lvert x\rvert^{n-1} + \lvert x\rvert^n$ $\le A(\lvert x\rvert + \dots + \lvert x\rvert^{n-1}) + \lvert x\rvert^n$ $\le A(\lvert x\rvert + \dots + \lvert x\rvert^{n-1} + \lvert x\rvert^n)$ (justified) $= A\frac{\lvert x\rvert(1-\lvert x\rvert^n)}{1-\lvert x\rvert}$ $\le A\frac{\lvert x\rvert}{1-\lvert x\rvert}$ (justified)	 M1 M1 M1 M1 M1 A1 (AG) (6 marks)
		(b)	$1 \le \frac{A\lvert\omega\rvert}{1-\lvert\omega\rvert}$ using $f(\omega) = 0$ $1 \le (A+1)\lvert\omega\rvert$ (with sign of $1 - \lvert\omega\rvert$ justified) $A + 1 \ge 1 \ge \lvert w\rvert$	M1 A1 (AG) B1 (AG) (3 marks)
		(c)	If $\lvert\omega\rvert > 1$, $0 = \omega^n f\left(\frac{1}{\omega}\right)$ $= 1 + a_{n-1}\omega + \dots + a_1\omega^{n-1} + \omega^n$ Inequalities continue to hold since $\lvert a_i\rvert \le A$ If $\lvert\omega\rvert = 1$, then $1 + A \ge 1 \ge \frac{1}{1+A}$ since $A > 0$	 M1 E1 E1 (3 marks)
	(ii)		$f(x) = x^5 - x^4 - \frac{100}{135}x^3 - \frac{91}{135}x^2 - \frac{126}{135}x + 1$ Use $A = 1$. Integer roots with $\frac{1}{2} \le \lvert\omega\rvert \le 2$ could only be ±1 or ±2 $f(\pm2) \ne 0$ because numerator is odd (or any valid justification) $f(1) = -\frac{182}{135} \ne 0$ $f(1) = 0$ $x = 1$ is the only integer root.	B1 M1 M1 E1 A1 A1 (6 marks)

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