STEP II, 2019, Q3

3 For any two real numbers x_1 and x_2, show that

$$|x_1 + x_2| \leqslant |x_1| + |x_2|.$$

Show further that, for any real numbers x_1, x_2, ..., x_n,

$$|x_1 + x_2 + \cdots + x_n| \leqslant |x_1| + |x_2| + \cdots + |x_n|.$$

(i) The polynomial f is defined by

$$\mathrm{f}(x) = 1 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + x^n$$

where the coefficients are real and satisfy $|a_i| \leqslant A$ for $i = 1, 2, \ldots, n-1$, where $A \geqslant 1$.

(a) If $|x| < 1$, show that

$$|\mathrm{f}(x) - 1| \leqslant \frac{A|x|}{1 - |x|}.$$

(b) Let ω be a real root of f, so that $\mathrm{f}(\omega) = 0$. In the case $|\omega| < 1$, show that

$$\frac{1}{1 + A} \leqslant |\omega| \leqslant 1 + A. \qquad (*)$$

(c) Show further that the inequalities $(*)$ also hold if $|\omega| \geqslant 1$.

(ii) Find the integer root or roots of the quintic equation

$$135x^5 - 135x^4 - 100x^3 - 91x^2 - 126x + 135 = 0.$$