STEP II, 2019, Q11 MS

In part (i), the numbers of ways of choosing the pairs can be found by checking the numbers of possible values for n_2 for each choice of n_1. A clear list of the possibilities for each case should then make generalised formulae for the cases $n_3 = 2n + 1$ and $n_3 = 2n$.

In part (ii), the possible combinations which lead to a triangle match those found in the first part of the question. There are $\binom{N-1}{2}$ possibilities for the shorter two rods if the length of the longest rod is known, so combining this with the answers to part (i) the probability can be calculated for each of the two cases to be considered.

In part (iii), the probability can be calculated by multiplying the probability in part (ii) for each possible length of the longest rod by the probability that that length is the longest of the three rods. Adding all of these together will result in the overall probability that the rods can form a triangle.

11	(i)		In both cases, award the M mark if all possible values of n_2 for at least 3 values of n_1 are identified.	
			$\boldsymbol{n_3 = 9}$ $n_1 = 1; n_2$ has no options $n_1 = 2; n_2 = 8$ $n_1 = 3; n_2 = 8, 7$ $n_1 = 4; n_2 = 8, 7, 6$ $n_1 = 5; n_2 = 8, 7, 6$ $n_1 = 6; n_2 = 8, 7$ $n_1 = 7; n_2 = 8$ $n_1 = 8; n_2$ has no options $\text{Total } = (1 + 2 + 3) \times 2 = 12$	M1
			$\boldsymbol{n_3 = 10}$ $n_1 = 1; n_2$ has no options $n_1 = 2; n_2 = 9$ $n_1 = 3; n_2 = 9, 8$ $n_1 = 4; n_2 = 9, 8, 7$ $n_1 = 5; n_2 = 9, 8, 7, 6$ $n_1 = 6; n_2 = 9, 8, 7$ $n_1 = 7; n_2 = 9, 8$ $n_1 = 8; n_2 = 9$ $n_1 = 0; n_2$ has no options	M1
			$\text{Total } = (1 + 2 + 3 + 4) \times 2 - 4 = 16$	A1 (both totals correct)
			$\boldsymbol{n_3 = 2n + 1}$ Total ways $= \big(1 + \cdots + (n-1)\big) \times 2$ (method mark may be implicit) $= (n-1)n$	M1 A1
			$\boldsymbol{n_3 = 2n}$ Total ways $= \big(1 + \cdots + (n-1)\big) \times 2 - (n-1)$ (method mark may be implicit) $= (n-1)^2$	M1 A1 (7 marks)
	(ii)		Total number of pairs is $\binom{N-1}{2} = \frac{1}{2}(N-1)(N-2)$	M1
			Justification for using first part of question	B1
			$\boldsymbol{N = 2n + 1}$ $\text{Prob} = \frac{(n-1)n}{\frac{1}{2}(2n)(2n-1)} = \frac{n-1}{2n-1}$	A1 (AG)
			$\boldsymbol{N = 2n}$ $\text{Prob} = \frac{(n-1)^2}{\frac{1}{2}(2n-1)(2n-2)} = \frac{n-1}{2n-1}$	A1 (4 marks)

	(iii)		Prob $= \sum_{n=1}^{M} \frac{n-1}{2n-1} \times \mathbb{P}(\text{largest rod is } 2n+1) + \sum_{n=1}^{M} \frac{n-1}{2n-1} \times \mathbb{P}(\text{largest rod is } 2n)$ $= \sum_{n=1}^{M} \frac{n-1}{2n-1}\left(\frac{\binom{2n}{2}}{\binom{2M+1}{3}} + \frac{\binom{2n-1}{2}}{\binom{2M+1}{3}}\right)$ $= \frac{6}{(2M+1)(2M)(2M-1)} \cdot \frac{1}{2}\sum_{n=1}^{M} \frac{n-1}{2n-1}(2n(2n-1)+(2n-1)(2n-2))$ (Use of formula for binomial coefficients with factorials cancelled) $= \frac{3}{M(2M+1)(2M-1)}\sum_{n=1}^{M}(n-1)(2n-1)$ Use of $\sum_1^K k^2 = \frac{1}{6}K(K+1)(2K+1)$ to simplify above $= \frac{3}{M(2M+1)(2M-1)}\left(\frac{1}{3}M(M+1)(2M+1) - 3 \times \frac{1}{2}M(M+1) + M\right)$ $= \frac{1}{2(2M+1)(2M-1)}(4M^2 - 3M - 1)$ $= \frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}$	M1 A1 (ft) M1 A1 M1 M1 M1 M1 A1 (9 marks)