STEP II, 2019, Q11

11 **(i)** The three integers n_1, n_2 and n_3 satisfy $0 < n_1 < n_2 < n_3$ and $n_1 + n_2 > n_3$. Find the number of ways of choosing the pair of numbers n_1 and n_2 in the cases $n_3 = 9$ and $n_3 = 10$.

Given that $n_3 = 2n+1$, where n is a positive integer, write down an expression (which you need not prove is correct) for the number of ways of choosing the pair of numbers n_1 and n_2. Simplify your expression.

Write down and simplify the corresponding expression when $n_3 = 2n$, where n is a positive integer.

(ii) You have N rods, of lengths 1, 2, 3, ... , N (one rod of each length). You take the rod of length N, and choose two more rods at random from the remainder, each choice of two being equally likely. Show that, in the case $N = 2n + 1$ where n is a positive integer, the probability that these three rods can form a triangle (of non-zero area) is

$$\frac{n-1}{2n-1}.$$

Find the corresponding probability in the case $N = 2n$, where n is a positive integer.

(iii) You have $2M+1$ rods, of lengths 1, 2, 3, ... , $2M+1$ (one rod of each length), where M is a positive integer. You choose three at random, each choice of three being equally likely. Show that the probability that the rods can form a triangle (of non-zero area) is

$$\frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}.$$

Note: $\sum_{k=1}^{K} k^2 = \frac{1}{6}K(K+1)(2K+1)$.