STEP II, 2019, Q1

1 Let $\mathrm{f}(x) = (x-p)\mathrm{g}(x)$, where g is a polynomial. Show that the tangent to the curve $y = \mathrm{f}(x)$ at the point with $x = a$, where $a \neq p$, passes through the point $(p, 0)$ if and only if $\mathrm{g}'(a) = 0$.

The curve C has equation

$$y = A(x-p)(x-q)(x-r),$$

where p, q and r are constants with $p < q < r$, and A is a non-zero constant.

(i) The tangent to C at the point with $x = a$, where $a \neq p$, passes through the point $(p, 0)$. Show that $2a = q + r$ and find an expression for the gradient of this tangent in terms of A, q and r.

(ii) The tangent to C at the point with $x = c$, where $c \neq r$, passes through the point $(r, 0)$. Show that this tangent is parallel to the tangent in part (i) if and only if the tangent to C at the point with $x = q$ does not meet the curve again.