# STEP II, 2018, Q8 MS

Making the substitution given reduces the differential equation into one for which it is easy to separate the variables. The two sides can then be integrated to find a general solution to the equation and then the boundary condition can be applied to find the required solution.

The differential equation in part (ii) is similar to the one from part (i), so a similar substitution should work (using a cube root rather than square root). The same process can then be followed as in part (i) to solve this differential equation.

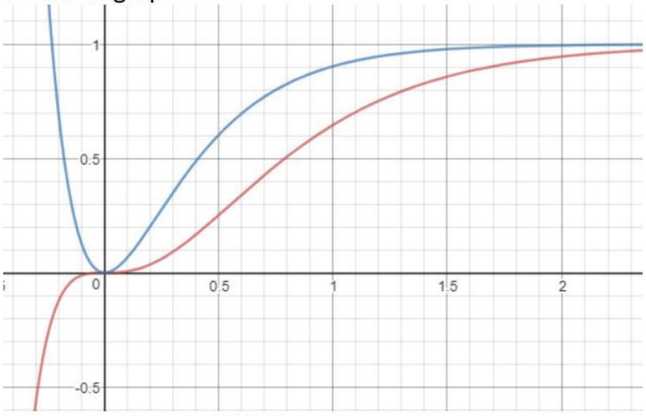
In part (iii), the information that $\alpha = \beta$ can be used to simplify the equations being considered and then it can be seen that the two curves will both approach an asymptote at $y = 1$. We also know that the curves both pass through the origin and the differential equations show that the curves should both have gradient 0 at the origin. All that remains is to deduce the relative positions of the two curves by considering the behaviour of the exponential function.

| | | |
|---|---|---|
| (i) | $\frac{dv}{dt} = \frac{1}{2}y^{-\frac{1}{2}} \times \frac{dy}{dt}$ | M1 |
| | $\frac{dy}{dt} = 2v\frac{dv}{dt}$ | A1 |
| | $2v\frac{dv}{dt} = \alpha v - \beta v^2$ | |
| | $\frac{dv}{dt} = \frac{1}{2}(\alpha - \beta v)$ | M1 |
| | $\int \frac{1}{\alpha - \beta v} dv = \int \frac{1}{2} dt$ | M1 |
| | $-\frac{1}{\beta}\ln\lvert\alpha - \beta v\rvert = \frac{1}{2}t + c$ | M1 |
| | $\alpha - \beta v = Ae^{-\frac{1}{2}\beta t}$ | M1 |
| | $v = \frac{1}{\beta}\left(\alpha - Ae^{-\frac{1}{2}\beta t}\right)$ | |
| | $y = \frac{1}{\beta^2}\left(\alpha - Ae^{-\frac{1}{2}\beta t}\right)^2$ | A1 |
| | $y_1 = \frac{\alpha^2}{\beta^2}\left(1 - e^{-\frac{1}{2}\beta t}\right)^2$ | A1 |
| (ii) | Use the substitution $v = y^{\frac{1}{3}}$: | M1 |
| | $\frac{dv}{dt} = \frac{1}{3}y^{-\frac{2}{3}} \times \frac{dy}{dt}$ | |
| | $3v^2\frac{dv}{dt} = \alpha v^2 - \beta v^3$ | |
| | $\frac{dv}{dt} = \frac{1}{3}\alpha - \frac{1}{3}\beta v$ | A1 |
| | $\int \frac{1}{\alpha - \beta v} dv = \int \frac{1}{3} dt$ | |
| | $-\frac{1}{\beta}\ln\lvert\alpha - \beta v\rvert = \frac{1}{3}t + c$ | |
| | $\alpha - \beta v = Ae^{-\frac{1}{3}\beta t}$ | |
| | $v = \frac{1}{\beta}\left(\alpha - Ae^{-\frac{1}{3}\beta t}\right)$ | A1 |
| | $y = \frac{1}{\beta^3}\left(\alpha - Ae^{-\frac{1}{3}\beta t}\right)^3$ | |
| | $y_2 = \frac{\alpha^3}{\beta^3}\left(1 - e^{-\frac{1}{3}\beta t}\right)^3$ | A1 |

(iii) If $\alpha = \beta$: **B1**

$y_1(x) = \left(1 - e^{-\frac{1}{2}\beta x}\right)^2$ and $y_2(x) = \left(1 - e^{-\frac{1}{3}\beta x}\right)^3$

Sketch of graphs:




Ignore anything to left of $y$-axis.

Both curves have a horizontal asymptote $y = 1$ **G1**

Both curves have gradient 0 as they pass through the origin **G1**

Both functions have decreasing gradient. **G1**

For positive values of $x$:

$0 > e^{-\frac{1}{3}\beta x} > e^{-\frac{1}{2}\beta x}$ **E1**

Therefore

$\left(1 - e^{-\frac{1}{3}\beta x}\right) < \left(1 - e^{-\frac{1}{2}\beta x}\right) < 1$ **E1**

$\left(1 - e^{-\frac{1}{3}\beta x}\right)^3 < \left(1 - e^{-\frac{1}{3}\beta x}\right)^2 < \left(1 - e^{-\frac{1}{2}\beta x}\right)^2$ **E1**

So the graph of $y_2$ should lie below the graph of $y_1$ **G1**