STEP II, 2018, Q6 MS

For the first part, note that 5 will certainly be a factor of the left-hand side in any case where $n \geq 5$. It then remains to check the other cases one at a time.

For part (ii), first explain how the two theorems show that there will not be any solutions if $n \geq 7$, by showing that $m > 4n$ and so there must be a prime factor of the right-hand side that cannot exist in the product on the left-hand side.

The remaining cases then need to be checked one at a time, noting that the individual numbers within the product can be split into their prime factorisation and then combined differently to form the right-hand side.

(i)	If $n \geq 5$ then $n! + 5 > 5$ and has 5 as a factor	**E1**
	Therefore the only possible solutions will have $n < 5$	**E1**
	The only pairs are therefore	
	$(2,7)$	**B1**
	$(3,11)$	**B1**
	$(4,29)$	**B1**
(ii)	If $n \geq 7$ then theorem 1 shows that $m > 4n$.	**E1**
	By theorem 2, there is a prime number between $2n$ and m, which must be a factor of $m!$	**E1**
	But that prime cannot be a factor of any of $1!, 3!, \cdots, (2n-1)!$	**E2**
	So it cannot be a factor of $1! \times 3! \times \cdots \times (2n-1)!$	**E1**
	Therefore there is a prime factor on the RHS that does not appear on the LHS.	**E1**
	Therefore the only pairs must have $n < 7$	**E1**
	$n = 1: m = 1$	**B1**
	$n = 2: m = 3$	**B1**
	$n = 3$: LHS=$3! \times 5!$	
	$$3! \times 5! = 5! \times 6 = 6!$$	
	So $m = 6$	**B1**
	$n = 4$: LHS= $3! \times 5! \times 7!$	**M1**
	$$3! \times 5! = 2 \times 3 \times 2 \times 3 \times 4 \times 5 = (2 \times 4) \times (3 \times 3) \times (2 \times 5)$$	
	So $m = 10$	**A1**
	$n = 5$: LHS = $3! \times 5! \times 7! \times 9!$	**E1**
	There must be two factors of 7 in the RHS, so $m \geq 14$	
	There will be no way of generating a factor of 11 for the RHS.	
	$n = 6$: LHS = $3! \times 5! \times 7! \times 9! \times 11!$	**E1**
	There must be two factors of 7 in the RHS, so $m \geq 14$	
	There will be no way of generating a factor of 13 for the RHS	**E1**