STEP II, 2018, Q5 MS

For the first part of the question note that $\ln(1+x)$ can be obtained by integrating $(1+x)^{-1}$ and so the required expansion can be found by integrating the binomial expansion term by term. Note also, that the integration produces a constant, which needs to be shown to be 0.

In part (ii), the series expansion of e^{ax} can be obtained by adjusting the series expansion of e^{x}. To evaluate the integral, substitute the series expansion for the e^{ax}, but leave the e^{-x} unchanged. The integration can then be completed term by term.

For part (iii) note that a substitution of $u = -\ln x$ will transform the integral into one that can be expressed in terms of the integral in part (ii), which then allows the result to follow.

(i) n^{th} term of expansion is $\frac{(-1)(-2)\ldots(-n)}{n!}(x)^n$

$$(1+x)^{-1} = \sum_{n=0}^{\infty}(-x)^n$$ **B1**

$$\int (1+x)^{-1}\,dx = \ln(1+x) + c$$ **B1**

E1

$$\int \sum_{n=0}^{\infty}(-x)^n\,dx = -\sum_{n=0}^{\infty}\frac{(-x)^{n+1}}{n+1} = -\sum_{n=1}^{\infty}\frac{(-x)^n}{n}$$ **B1**

AG

(ii)

$$e^{-ax} = \sum_{n=0}^{\infty}\frac{(-ax)^n}{n!}$$ **B1**

$$\frac{(1-e^{-ax})e^{-x}}{x} = -\left(\sum_{n=1}^{\infty}\frac{(-a)^n}{n!}x^{n-1}\right)e^{-x}$$ **M1**

Let

$$I_n = \int_0^{\infty} x^n e^{-x}\,dx$$

$$u = x^n \qquad \frac{dv}{dx} = e^{-x}$$ **M1**

A1

$$\frac{du}{dx} = nx^{n-1} \qquad v = -e^{-x}$$

$$I_n = [-x^n e^{-x}]_0^{\infty} + n\int_0^{\infty} x^{n-1}e^{-x}\,dx$$ **M1**

$$I_n = nI_{n-1}$$ **M1**

$$I_0 = \int_0^{\infty} e^{-x}\,dx = [-e^{-x}]_0^{\infty} = 1$$ **M1**

Therefore $I_n = n!$ **A1**

So **A1**

$$\int_0^{\infty} -\left(\sum_{n=1}^{\infty}\frac{(-a)^n}{n!}x^{n-1}\right)e^{-x}\,dx = -\left(\sum_{n=1}^{\infty}\frac{(-a)^n}{n}\right) = \ln(1+a)$$ **AG**

(by part (i))

(iii) Let $u = -\ln x$ **M1**

Then $x = e^{-u}$ and $\frac{dx}{du} = -e^{-u}$ **M1**

Change limits:

$x = 1$ becomes $u = 0$

$x = 0$ becomes $u = \infty$ **B1**

$$\int_0^1 \frac{x^p - x^q}{\ln x}\,dx = \int_{\infty}^{0}\frac{(e^{-pu}-e^{-qu})}{-u}(-e^{-u})du$$ **M1**

A1

$$= -\int_0^{\infty}\frac{(1-e^{-qu})-(1-e^{-pu})}{u}(e^{-u})du$$ **M1**

$$= -\ln(1+q) + \ln(1+p)$$

$$= \ln\left(\frac{1+p}{1+q}\right)$$ **A1**