STEP II, 2018, Q4 MS

For the first part, note that two of the terms in the left-hand side of the equation have a coefficient of 1 and two have a coefficient of 3. Applying the given identity to each of these pairs gives a common factor of $\cos\frac{5x}{2}$. The equation can therefore be factorised and then another application of the given identity will allow the full set of roots to be found.

The identity given at the start of the question can be applied to the first two terms of the left-hand side of the equation in part (ii) and the double angle formula can be applied to the $\cos 2x$. This then leads to an equation that can easily be factorised to show the required result. The range of possible values needs to be considered when considering the case where $\cos x = \cos y$.

For the final part a similar process to part (ii) can be used to create a quadratic function of $\cos\frac{1}{2}(x+y)$. Completing the square or considering a discriminant then allows the solutions to be found.

(i)	$\cos x + \cos 4x = 2\cos\frac{5}{2}x\cos\frac{3}{2}x$ and $\cos 2x + \cos 3x = 2\cos\frac{5}{2}x\cos\frac{1}{2}x$	**M1**
	$2\cos\frac{5}{2}x\cos\frac{3}{2}x + 6\cos\frac{5}{2}x\cos\frac{1}{2}x = 0$, so $2\cos\frac{5}{2}x\left(\cos\frac{3}{2}x + 3\cos\frac{1}{2}x\right) = 0$	**M1**
	Therefore $\cos\frac{5}{2}x = 0$ or $\cos\frac{3}{2}x + 3\cos\frac{1}{2}x = 0$	**A1**
	$\cos\frac{5}{2}x = 0$ gives $x = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}$ or $\frac{9\pi}{5}$	**B1 B1**
	If $\cos\frac{3}{2}x + 3\cos\frac{1}{2}x = 0$, then:	**M1**
	$\left(\cos\frac{3}{2}x + \cos\frac{1}{2}x\right) + 2\cos\frac{1}{2}x = 0$	
	$2\cos x\cos\frac{1}{2}x + 2\cos\frac{1}{2}x = 0$	
	$2\cos\frac{1}{2}x(\cos x + 1) = 0$	
	$\cos\frac{1}{2}x = 0$ or $\cos x = -1$, both of which give no new solutions to the equation.	**A1**
(ii)	$\cos(x-y) + \cos(x+y) = 2\cos x\cos y$	**M1**
	$2\cos x\cos y - 2\cos^2 x + 1 = 1$	**M1**
	$2\cos x(\cos y - \cos x) = 0$	**M1**
	Therefore either $\cos x = \cos y$, which can only be the case if $x = y$ since $0 \le x \le \pi$ and $0 \le y \le \pi$	**E1**
	Or $\cos x = 0$, so $x = \frac{\pi}{2}$	**A1**
(iii)	$2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$	**M1**
	$-\left(2\cos^2\frac{1}{2}(x+y) - 1\right) = \frac{3}{2}$	**M1**
	$4\cos^2\frac{1}{2}(x+y) - 4\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y) + 1 = 0$	**M1**
	$\left(2\cos\frac{1}{2}(x+y) - \cos\frac{1}{2}(x-y)\right)^2 + 1 - \cos^2\frac{1}{2}(x-y) = 0$	**M1**
	$\left(2\cos\frac{1}{2}(x+y) - \cos\frac{1}{2}(x-y)\right)^2 + \sin^2\frac{1}{2}(x-y) = 0$	**M1**
	Therefore, since both terms are ≥ 0, they must both be equal to 0.	**M1**
	For $0 \le x \le \pi$ and $0 \le y \le \pi$, $\sin^2\frac{1}{2}(x-y) = 0$ only when $x = y$	**M1**
	Therefore $2\cos x = 1$, so $x = \frac{\pi}{3}$ and $y = \frac{\pi}{3}$	**A1**