STEP II, 2018, Q3 MS

The differentiation for the first part of the question can be achieved by applying the chain rule. Consideration of the sine function within the interval then allows the range to be determined. Consideration of the gradient function then allows the graph to be sketched - it can be seen from the symmetry of the sine function that $f'(x) = f'\left(\frac{1}{2}\pi\right) - x$, which means that the graph must have rotational symmetry about the point where $x = \frac{1}{4}\pi$.

A sketch of a rotationally symmetric function is a helpful way of demonstrating the second part as it allows the distances that must be equal to be identified clearly. It is important to show clearly that the result works in both the if and only if directions. For the final question in this part, note that the sections of the graph above the axis must exactly match those below the axis, so the area must be 0.

For part (iii), begin by showing that the equation from part (ii) holds for this function. Once the rotational symmetry has been demonstrated it follows that the area of any interval with the centre of rotation in the centre will be equal to the area of a rectangle over the same interval passing through the centre of rotation.

(i) $f(x) = (1 + \tan x)^{-1}$ **M1**

$f'(x) = -(1 + \tan x)^{-2} \sec^2 x$ **A1**

$f'(x) = -\frac{1}{(1+\tan x)^2 \cos^2 x}$

$= -\frac{1}{(\sin x + \cos x)^2}$ **M1**

$= -\frac{1}{\sin^2 x + \cos^2 x + 2\sin x \cos x}$ **A1**

$= -\frac{1}{1+\sin 2x}$ **AG**

Within the given domain, $0 \leq \sin 2x \leq 1$, so $-1 \leq f'(x) \leq \frac{1}{2}$ **B1**

Sketch of graph should have the following features:

Decreasing function **G1**

Points (0,1) and $\left(\frac{\pi}{2}, 0\right)$ **G1**

Point of inflexion at $x = \frac{\pi}{4}$ **G1**

All other features correct **G1**

(ii) If the point $(x, g(x))$ is rotated through 180 degrees about the point (a, b) then the image will be at the point $(a + (a - x), b + (b - g(x)))$. **E1**

Therefore, if the curve has rotational symmetry of order 2 about the point (a, b), then $g(2a - x) = 2b - g(x)$, so $g(x) + g(2a - x) = 2b$ **E1**

Similarly, if $g(x) + g(2a - x) = 2b$, then any pair of points that are centred horizontally on the point (a, b) will also be centred vertically on the point (a, b), **E1**

which means that the curve will have rotational symmetry about that point. **E1**

$\int_{-1}^{1} g(x)\, dx = 0$ **B1**

(iii) Since $\tan\left(\frac{\pi}{2} - x\right) = \cot x$, **B1**

$f\left(\frac{\pi}{2} - x\right) = \frac{1}{1+\cot^k x}$ **M1**

$= \frac{\tan^k x}{\tan^k x + 1} = 1 - f(x)$ **M1**

Therefore $f(x) + f\left(2\left(\frac{\pi}{4}\right) - x\right) = 2\left(\frac{1}{2}\right)$

So the curve has rotational symmetry of order 2 about the point $\left(\frac{\pi}{4}, \frac{1}{2}\right)$ **A1**

The area under the curve over any interval centred on $x = \frac{\pi}{4}$, will therefore have the same area as a rectangle of the same width and height $\frac{1}{2}$. **M1**

Therefore $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1+\tan^k x}\, dx = \left(\frac{\pi}{3} - \frac{\pi}{6}\right) \times \frac{1}{2} = \frac{\pi}{12}$ **A1**