STEP II, 2018, Q3

3 **(i)** Let

$$\mathrm{f}(x) = \frac{1}{1+\tan x}$$

for $0 \leqslant x < \frac{1}{2}\pi$.

Show that $\mathrm{f}'(x) = -\dfrac{1}{1+\sin 2x}$ and hence find the range of $\mathrm{f}'(x)$.

Sketch the curve $y = \mathrm{f}(x)$.

(ii) The function $\mathrm{g}(x)$ is continuous for $-1 \leqslant x \leqslant 1$.

Show that the curve $y = \mathrm{g}(x)$ has rotational symmetry of order 2 about the point (a, b) on the curve if and only if

$$\mathrm{g}(x) + \mathrm{g}(2a - x) = 2b\,.$$

Given that the curve $y = \mathrm{g}(x)$ passes through the origin and has rotational symmetry of order 2 about the origin, write down the value of

$$\int_{-1}^{1} \mathrm{g}(x)\,\mathrm{d}x\,.$$

(iii) Show that the curve $y = \dfrac{1}{1+\tan^k x}$, where k is a positive constant and $0 < x < \frac{1}{2}\pi$, has rotational symmetry of order 2 about a certain point (which you should specify) and evaluate

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1+\tan^k x}\,\mathrm{d}x\,.$$