# STEP II, 2018, Q2 MS

The point with $x$-coordinate $tx_1 + (1-t)x_2$ is a point within the range $(x_1, x_2)$ (for $0 < t < 1$), and $tf(x_1) + (1-t)f(x_2)$ is the $y$-coordinate of a point on the chord joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Therefore, for any position between the two endpoints on the curve, the inequality is comparing the point on the curve with the point on the chord joining the two points. The sketch therefore needs to show the chord entirely below the curve. The final part of the introductory section can be shown either with a proof by contradiction or by arguing that $f''(x) < 0$ means that the gradient is always decreasing within the interval.

Part (i) requires choosing values for $x_1$ and $x_2$ so that the inequality can be applied and then applying this process multiple times to reach the required result. In each case the choice of $x_1$ and $x_2$ need to be made so that they lie within the range for which applying the inequality is valid.

Parts (ii) and (iii) both follow from the result of part (i), but it is important to check that the function being used is concave in the relevant range which needs to be stated clearly. In part (ii) the result follows immediately, whereas the final part requires some manipulation of logarithms to reach the final form of the relationship.

Sketch showing the curve and chord with the chord entirely below the curve and $f(x_1) < f(x_2)$ **E1**

$tx_1 + (1-t)x_2$ identified as a value in the range $(x_1, x_2)$ **E1**

$\left(tx_1 + (1-t)x_2, tf(x_1) + (1-t)f(x_2)\right)$ identified as the point on the chord. **E1**

If $f''(x) < 0$ for $a < x < b$ then the gradient of the curve $y = f(x)$ must be decreasing as $x$ increases. **E1**

Suppose that a function $f(x)$ satisfies $f''(x) < 0$ for $a < x < b$, but is not concave for $a < x < b$. Then there must be points $x_1 < x_2$ and a value $t$, $0 < t < 1$ such that

$tf(x_1) + (1-t)f(x_2) > f(tx_1 + (1-t)x_2)$

The gradient at $x = tx_1 + (1-t)x_2$ must be less than the gradient of the chord joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$, and so the curve $y = f(x)$ must continue to have a gradient of this value or less. The curve therefore cannot pass through $(x_2, f(x_2))$. Therefore, it must be the case that a function satisfying $f''(x) < 0$ for $a < x < b$ is concave for $a < x < b$. **E1**

(i) Let $x_1 = \frac{2u+v}{3}, x_2 = \frac{v+2w}{3}$ and $t = \frac{1}{2}$ **M1**

Then, since $f(x)$ is concave for $a < x < b$: **A1**

$\frac{1}{2}f\left(\frac{2u+v}{3}\right) + \frac{1}{2}f\left(\frac{v+2w}{3}\right) \leq f\left(\frac{u+v+w}{3}\right)$

Setting $x_1 = u, x_2 = v$ and $t = \frac{2}{3}$ gives: **B1**

$\frac{2}{3}f(u) + \frac{1}{3}f(v) \leq f\left(\frac{2u+v}{3}\right)$

Similarly, setting $x_1 = v, x_2 = w$ and $t = \frac{1}{3}$ gives: **B1**

$\frac{1}{3}f(v) + \frac{2}{3}f(w) \leq f\left(\frac{v+2w}{3}\right)$

Therefore: **M1**

$f\left(\frac{u+v+w}{3}\right) \geq \frac{1}{2}f\left(\frac{2u+v}{3}\right) + \frac{1}{2}f\left(\frac{v+2w}{3}\right)$ **A1 AG**

$\geq \frac{1}{2}\left(\frac{2}{3}f(u) + \frac{1}{3}f(v)\right) + \frac{1}{2}\left(\frac{1}{3}f(v) + \frac{2}{3}f(w)\right) = \frac{f(u)+f(v)+f(w)}{3}$

(ii) If $f(x) = \sin x$, then $f''(x) = -\sin x$ and $f''(x) < 0$ for $0 < x < \pi$. **B1**

Therefore $f(x)$ is concave for $0 < x < \pi$. **E1**

$0 < A, B, C < \pi$ and $A + B + C = \pi$, therefore, by (i): **M1**

$\sin\frac{\pi}{3} \geq \frac{\sin A + \sin B + \sin C}{3}$

$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$ **A1 AG**

(iii) If $f(x) = \ln(\sin x)$, then $f'(x) = \cot x$ **M1**

$f''(x) = -\operatorname{cosec}^2 x$ **A1**

Therefore $f''(x) < 0$ for $0 < x < \pi$ and so $f(x)$ is concave for $0 < x < \pi$ **E1**

Therefore: **M1**

$\ln\left(\sin\frac{\pi}{3}\right) \geq \frac{\ln(\sin A) + \ln(\sin B) + \ln(\sin C)}{3}$

$3\ln\left(\frac{\sqrt{3}}{2}\right) \geq \ln(\sin A \times \sin B \times \sin C)$

$\sin A \times \sin B \times \sin C \leq \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}$ **A1 AG**