STEP II, 2018, Q2

2 A function $\mathrm{f}(x)$ is said to be *concave* for $a < x < b$ if

$$t\,\mathrm{f}(x_1) + (1-t)\,\mathrm{f}(x_2) \leqslant \mathrm{f}\big(tx_1 + (1-t)x_2\big)\,,$$

for $a < x_1 < b$, $a < x_2 < b$ and $0 \leqslant t \leqslant 1$.

Illustrate this definition by means of a sketch, showing the chord joining the points $\big(x_1, \mathrm{f}(x_1)\big)$ and $\big(x_2, \mathrm{f}(x_2)\big)$, in the case $x_1 < x_2$ and $\mathrm{f}(x_1) < \mathrm{f}(x_2)$.

Explain why a function $\mathrm{f}(x)$ satisfying $\mathrm{f}''(x) < 0$ for $a < x < b$ is concave for $a < x < b$.

(i) By choosing t, x_1 and x_2 suitably, show that, if $\mathrm{f}(x)$ is concave for $a < x < b$, then

$$\mathrm{f}\Big(\frac{u+v+w}{3}\Big) \geqslant \frac{\mathrm{f}(u) + \mathrm{f}(v) + \mathrm{f}(w)}{3}\,,$$

for $a < u < b$, $a < v < b$ and $a < w < b$.

(ii) Show that, if A, B and C are the angles of a triangle, then

$$\sin A + \sin B + \sin C \leqslant \frac{3\sqrt{3}}{2}\,.$$

(iii) By considering $\ln(\sin x)$, show that, if A, B and C are the angles of a triangle, then

$$\sin A \times \sin B \times \sin C \leqslant \frac{3\sqrt{3}}{8}\,.$$