STEP II, 2018, Q1 MS

The first result can be shown by substituting k^{-1} into the quartic expression.

It then follows for part (i) that the only way to achieve one distinct root is for that root to be either 1 or -1. In either case the factorised form of the quartic can then be considered to find the values of a and b.

Similarly, for three distinct roots, there must be one pair (k, k^{-1}) along with either 1 or -1. Substitution of 1 or -1 into the quartic then leads to the required relationships.

For part (iii) note that the case $b = 2a - 2$ corresponds to the case where there is a root of -1 and it can be seen that it must be a repeated root. The other factor is therefore a quadratic, which can then be solved.

Finally, the conditions for there to be three roots can be found by considering the discriminant of the quadratic (and the corresponding one for the other case). It is also necessary to confirm that this quadratic does not repeat the root of either 1 or -1 depending on which case is being considered.

	Substitute $x = k^{-1}$ into the quartic expression:	M1
	$$k^{-4} + ak^{-3} + bk^{-2} + ck^{-1} + 1 = \frac{1 + ak + bk^2 + ak^3 + k^4}{k^4}$$	
	Since k cannot be 0 and the numerator is equal to 0 (since k is a root of the equation), k^{-1} must also be a solution to the equation.	E1
(i)	For there to be only one distinct root, the root must be either 1 or -1	
	If the root is 1 then $a = -4, b = 6$	B1
	If the root is -1 then $a = 4, b = 6$	B1
(ii)	For there to be three distinct roots there must be one repeated root (which must be either 1 or -1).	E1
	If the repeated root is $x = 1$ then:	M1
	$1 + a + b + a + 1 = 0$	
	Therefore $b = -2a - 2$	A1 AG
	If the repeated root is $x = -1$ then:	M1
	$1 - a + b - a + 1 = 0$	
	Therefore $b = 2a - 2$	A1 AG
(iii)	$b = 2a - 2$ corresponds to the case where the repeated root is -1.	
	$x^4 + ax^3 + bx^2 + ax + 1 = (x + 1)(x^3 + (a - 1)x^2 + (a - 1)x + 1)$	
	$(x + 1)$ is a factor of $(x^3 + (a - 1)x^2 + (a - 1)x + 1)$	
	$x^4 + ax^3 + bx^2 + ax + 1 = (x^2 + 2x + 1)(x^2 + kx + 1)$	M1
	Comparing coefficients of x^3:	A1
	$a = k + 2$	
	Therefore the other roots are	M1
	$\frac{(2 - a) \pm \sqrt{(a - 2)^2 - 4}}{2}$	A1
	In the case where $b = 2a - 2$:	
	For all three roots to be real, $(a - 2)^2 - 4 > 0$	M1
		A1
	$a^2 > 4a = 2b + 4$	
	In the case where $b = -2a - 2$, the quadratic will have $a = k - 2$	M1
	Therefore $(a + 2)^2 - 4 > 0$ for three roots	A1
	The quadratic factors in the two cases are both of the form $x^2 + kx + 1$. They	M1
	must have roots that are not ± 1.	A1
	$\frac{k \pm \sqrt{k^2 - 4}}{2} = \pm 1$ if $k^2 - 4 = (k \pm 2)^2$,	
	$(k \pm 2)^2 - (k + 2)(k - 2) = 0$, so $k = \pm 2$.	
	Therefore in neither of the two cases investigated does the quadratic equation have solutions of ± 1	
	Therefore	A1
	$(b + 2)^2 = 4a^2$	
	and	
	$a^2 > 2b + 4$	
	Are necessary and sufficient conditions for (*) to have exactly three distinct real roots.	