STEP II, 2018, Q12 MS

First note that the only winning sequence is h heads in a row and the probability of this can be found easily. The expected winnings can then be expressed as a function of h (E_h). By considering the value of $\frac{E_{h+1}}{E_h}$ it can be shown that the expected winnings increases until $h = N$, remain the same for the next case and then decreases thereafter.

For the second part there are multiple sequences that lead to a win. Begin with a sequence of h heads and then consider adding tails to any of the h positions before those heads (only 1 tail can be placed in each position). The number of ways of winning with a total of t tails in the sequence can therefore be seen to be $\binom{h}{t}$. The sum of these probabilities can then be seen to be a binomial expansion and can therefore be simplified. An expression for the expected winnings can therefore be found. The case where $N = 2$ leads to a function of the form of the previous part, so the point at which the maximum value occurs can be written down immediately. Taking logarithms of this maximum value allows it to be shown that the value is very close to 3^1.

(i)	I will win if there are h consecutive heads and lose otherwise.	M1
	$P(h\ consecutive\ heads) = p^h \left[= \left(\frac{N}{N+1}\right)^h\right]$	
	$Expected\ winnings = p^h h \left[= \left(\frac{N}{N+1}\right)^h h\right]$	A1
	Let E_h be the expected winnings when the value h is chosen.	M1
	$$\frac{E_{h+1}}{E_h} = \left(\frac{N}{N+1}\right)\left(\frac{h+1}{h}\right) = \frac{Nh+N}{Nh+h}$$	A1
	Therefore $\frac{E_{h+1}}{E_h} > 1$ if $h < N$	M1
	And $\frac{E_{h+1}}{E_h} < 1$ if $h > N$	M1
	So as h increases, the values of E_h increase until $h = N$, the value then remains the same for $h = N + 1$ and decreases thereafter. So I can maximise my winnings by choosing $h = N$	A1
(ii)	Possible sequences that lead to a win are:	
	All heads: Probability: $\left(\frac{N}{N+1}\right)^h$	
	There are h positions available (one before each of the heads) where at most one tail can be placed.	B1
	1 tail can be placed in any of the h positions, so the probability of a sequence containing just one tail is $\binom{h}{1}\left(\frac{N}{N+1}\right)^h\left(\frac{1}{N+1}\right)^1$	M1
	Similarly, for any other number of tails, $t \leq h$, the probability of a winning sequence containing that number of tails will be $\binom{h}{t}\left(\frac{N}{N+1}\right)^h\left(\frac{1}{N+1}\right)^t$	M1
	Therefore the probability that I win is	M1
	$$\sum_{t=0}^{h}\binom{h}{t}\left(\frac{N}{N+1}\right)^h\left(\frac{1}{N+1}\right)^t = \left(\frac{N}{N+1}\right)^h\sum_{t=0}^{h}\binom{h}{t}\left(\frac{1}{N+1}\right)^t$$	A1
	As the sum in the expression on the right is a binomial expansion it can be	M1
	rewritten as $\left(\frac{1}{N+1}+1\right)^h$	A1
	The probability that I win is therefore	
	$$\left(\frac{N}{N+1}\right)\left(\frac{1}{N+1}+1\right)^h = \frac{N^h(1+N+1)^h}{(N+1)^{2h}} = \frac{N^h(N+2)^h}{(N+1)^{2h}}$$	
	So my expected winnings are $\frac{hN^h(N+2)^h}{(N+1)^{2h}}$	A1 AG
	In the case $N = 2$, the expected winnings are $h\left(\frac{8}{9}\right)^h$	
	The maximum value is when $h = 8$ or $h = 9$ and has a value of $\frac{8^9}{9^8}$	B1
	$\log_3\left(\frac{8^9}{9^8}\right) = 9\log_3 8 - 8\log_3 9$	M1
	$= 27\log_3 2 - 16$	M1
	$\approx 27(0.63) - 16 = 1.01$	M1
	Therefore $\frac{8^9}{9^8} \approx 3^{1.01} \approx 3$	A1 AG