STEP II, 2018, Q12

12 In a game, I toss a coin repeatedly. The probability, p, that the coin shows Heads on any given toss is given by

$$p = \frac{N}{N+1},$$

where N is a positive integer. The outcomes of any two tosses are independent.

The game has two versions. In each version, I can choose to stop playing after any number of tosses, in which case I win £H, where H is the number of Heads I have tossed. However, the game may end before that, in which case I win nothing.

(i) In version 1, the game ends when the coin first shows Tails (if I haven't stopped playing before that).

I decide from the start to toss the coin until a total of h Heads have been shown, unless the game ends before then. Find, in terms of h and p, an expression for my expected winnings and show that I can maximise my expected winnings by choosing $h = N$.

(ii) In version 2, the game ends when the coin shows Tails on two *consecutive* tosses (if I haven't stopped playing before that).

I decide from the start to toss the coin until a total of h Heads have been shown, unless the game ends before then. Show that my expected winnings are

$$\frac{hN^h(N+2)^h}{(N+1)^{2h}}.$$

In the case $N = 2$, use the approximation $\log_3 2 \approx 0.63$ to show that the maximum value of my expected winnings is approximately £3.