STEP II, 2018, Q10 MS

Finding an expression for the length of the string at time t allows the speed of the point on the string to be determined. The differential equation can then be set up by adding the speed of the ant to the speed of the point on the string. The next result can then be verified by applying the quotient rule to perform the differentiation.

Once the differential equation has been verified, integration leads to a relationship between x and t, which then leads to the required result.

For the journey back, the differential equation needs to be changed so that the speed of the ant is subtracted rather than added. The differential equation can then be rewritten in a manner similar to the first part of the question and solved.

At time t the string will have a length of $a + ut$	M1
The speed of the point on the string will therefore be $\frac{xu}{a+ut}$	A1
$\frac{dx}{dt} = \frac{xu}{a+ut} + v$	B1
$\frac{d}{dt}\left(\frac{x}{a+ut}\right) = \frac{(a+ut)\frac{dx}{dt} - xu}{(a+ut)^2}$	M1 A1
$= \frac{xu + v(a+ut) - xu}{(a+ut)^2} = \frac{v}{a+ut}$	M1 A1 AG
$\frac{x}{a+ut} = \int \frac{v}{a+ut} dt$	M1
$\frac{x}{a+ut} = \frac{v}{u}\ln\lvert C(a+ut)\rvert$	A1
At $t = 0, x = 0$:	
$0 = \frac{v}{u}\ln aC$	M1
Therefore $C = \frac{1}{a}$	A1
At $t = T, x = a + uT$:	
$\frac{a+uT}{a+uT} = \frac{v}{u}\ln\left\lvert\frac{1}{a}(a+uT)\right\rvert$	M1
$1 + \frac{uT}{a} = e^k$	
where $k = u/v$.	
$uT = a(e^k - 1)$	A1 AG
For the journey back:	
$\frac{dx}{dt} = \frac{xu}{a+ut} - v$	M1
$\frac{d}{dt}\left(\frac{x}{a+ut}\right) = -\frac{v}{a+ut}$	M1
Therefore	
$\frac{x}{a+ut} = -\frac{v}{u}\ln\lvert C(a+ut)\rvert$	A1
At $t = T, x = a + uT$:	
$\frac{a+uT}{a+uT} = -\frac{v}{u}\ln\lvert C(a+uT)\rvert$	M1
Therefore:	
$C(a+uT) = e^{-k}$	A1
Solve for $x = 0$:	
$0 = -\frac{v}{u}\ln\lvert C(a+ut)\rvert$	M1
Therefore	
$C(a+ut) = 1$	
$e^{-k}(a+ut) = a + uT$	
$t = \frac{(a+uT)e^k - a}{u}$	
Therefore the time for the journey back is:	
$\frac{(a+uT)e^k - a}{u} - \frac{a(e^k-1)}{u} = Te^k$	A1 CAO