STEP II, 2017, Q9 MS

Before doing anything else in a question like this you need to draw a BIG diagram with all relevant forces labelled.
With nearly all questions like this it is a matter of setting up equations by resolving forces and taking moments. The main tactical decision is about in which direction to resolve forces and about which point to take moments. Because the equation we are trying to find in part (i) does not involve the weight or the normal reaction force of the cylinder and the ground there is a strong indication that we will be resolving horizontally. Focussing on just one cylinder we get:

$$F + F_1 \cos\theta = R \sin\theta \qquad (1)$$

where F_1 is the friction between the plank and the cylinder.
We also want to find a point that is in line with these forces so that when we take moments about that point the forces will not feature. A natural point to choose is the centre of the cylinder. Taking moments about this point gives:

$$F.r = F_1.r$$

So $F = F_1$. Substituting this into equation (1) gives the required result.
There are several ways in which an inequality can arise in mechanics, but in questions on friction a good possibility is using the fact that $F_1 \leq \mu R$. Combining this with the above results gets the required inequality.

Part (ii) does bring in the normal reaction with the floor, so resolving vertically will be a useful tool. For the cylinder this gives:

$$W \ = \ N - R\cos\theta - F_1 \sin\theta$$

where W is the weight of the cylinder. Unfortunately, we do not want W in our expression, but we do need to bring in a k. The easiest way to do this is to resolve vertically for the plank:

$$kW \ = \ 2R\cos\theta \ + \ 2F\sin\theta$$

Combining these two expressions with the result from part (i) and the fact that $\cos^2\theta + \sin^2\theta = 1$ leads to the required result.
If there is no slipping then $F \leq \mu N$. Substituting in the result we have just obtained turns this into something that can be rearranged into something similar to the last result in part (i):

$$2k\sin\theta \leq (k \ + \ 2)(1 + \cos\theta) \qquad (2)$$

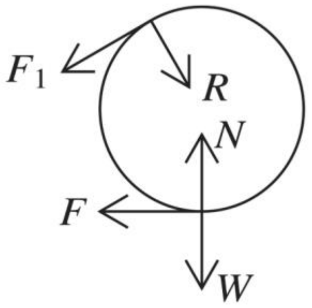
It is always important when dealing with these algebraic expressions to try to make links. We can use the final result in part (i) to show that (by multiplying by k):

$$2k\sin\theta \leq k(1+\cos\theta)$$

But $k(1+\cos\theta)$ must be less than $(k+2)(1+\cos\theta)$, therefore as long as the inequality from (i) is true then inequality (2) will be satisfied.
Sometimes it is important to see the thrust of a question. The first two parts have been steering you towards the idea that the important inequality is $2\sin\theta \leq 1+\cos\theta$. Our task is to now turn this into an inequality involving only $\sin\theta$. It is tempting to subtract 1 from both sides and square, as that will lead to an inequality involving only sines. However that is technically flawed: we cannot easily square expressions which are sometimes positive and sometimes negative [i.e., it is not generally true that $a<b$ leads to $a^2<b^2$; for instance, consider $-2<1$]. It is better to square up the original expression as it is, in the context of the question, never negative. After a bit of manipulation, it becomes:

$$0 \leq (5\cos\theta - 3)(\cos\theta + 1)$$

From this it can be deduced that $\cos\theta \geq \frac{3}{5}$, and in turn a graphical argument leads to the required result.
This then needs to be related to the physical situation. By finding an appropriate right-angled triangle you can show that $\sin\theta = \frac{r-a}{r}$, which leads to the required result.

(i) 		
$O \circlearrowleft$ $F.r = F_1.r$ $\Rightarrow F = F_1$	B1	For correct moment equation.
Res. $\leftrightarrow$ $F + F_1\cos\theta = R\sin\theta$	B1	For resolving horizontally for one cylinder.
Together give $R\sin\theta = F(1+\cos\theta)$	AG	
Since $F_1 \le \mu R$, with $\mu = \frac{1}{2}$,	M1	Use of the Friction law
it follows that $\frac{F}{R} \le \frac{1}{2} \Rightarrow \frac{\sin\theta}{1+\cos\theta} \le \frac{1}{2}$	A1	Combining with previous answer
i.e. $2\sin\theta \le 1+\cos\theta$	AG Subtotal: 4	
(ii) Res. $\uparrow$ for RH cylinder $W = N - R\cos\theta - F_1\sin\theta$	B1	F_1 might correctly be replaced with F.
Res. $\uparrow$ for plank $kW = 2R\cos\theta + 2F\sin\theta$	B1	
Eliminating W: $k(N - R\cos\theta - F\sin\theta) = 2R\cos\theta + 2F\sin\theta$	M1	For eliminating W
$N = R\cos\theta\left(\frac{2}{k}+1\right) + F\sin\theta\left(\frac{2}{k}+1\right)$	A1	For correct rearrangement for N
$N = \left(\frac{2}{k}+1\right)\left(\frac{1+\cos\theta}{\sin\theta}.\cos\theta + \sin\theta\right)F$	M1	For use of $R = \left(\frac{1+\cos\theta}{\sin\theta}\right)F$
$N = \left(\frac{2}{k}+1\right)\left(\frac{\cos\theta + \cos^2\theta + \sin^2\theta}{\sin\theta}\right)F$	B1	Obtaining $\cos^2\theta + \sin^2\theta$
$= \left(\frac{2}{k}+1\right)\left(\frac{\cos\theta + 1}{\sin\theta}\right)F$	AG	
For no slipping at the ground, $F \le \mu N$	M1	Using Friction equation
$\Rightarrow F \le \frac{1}{2}\left(\frac{2}{k}+1\right)\left(\frac{\cos\theta+1}{\sin\theta}\right)F$	A1	Using previous part
ie. $2k\sin\theta \le (k+2)(1+\cos\theta)$	A1	Rearranging into a "useful" form.

However, we already have that $2k\sin\theta \le k(1+\cos\theta) \le (k+2)(1+\cos\theta)$ so there are no extra restrictions on θ.	**E1** **Subtotal: 10**	Properly justified
(iii) $4\sin^2\theta \le 1 + 2\cos\theta + \cos^2\theta$ $4(1-\cos^2\theta) \le 1 + 2\cos\theta + \cos^2\theta$ $0 \le 5\cos^2\theta + 2\cos\theta - 3$ $0 \le (5\cos\theta - 3)(\cos\theta + 1)$ Since $\cos\theta \ge 0$ we have $\cos\theta \ge \frac{3}{5}$ For appropriate angles $\cos\theta$ is decreasing and $\sin\theta$ is increasing. Therefore $\sin\theta \le \frac{4}{5}$ $\sin\theta = \dfrac{r-a}{r}$ So $5r - 5a \le 4r$ $r \le 5a$	**M1** **M1** **A1** **E1** **AG** **B1** **M1** **AG** **Subtotal: 6**	Squaring up an appropriate trig inequality Creating and simplifying quadratic inequality in one trig ratio A graphical argument is perfectly acceptable here. N.b It is possible that inequalities like $2s - 1 \le c$ are squared. If this is done without justifying that both sides are positive then withhold this final **E1**. Combining with previous result