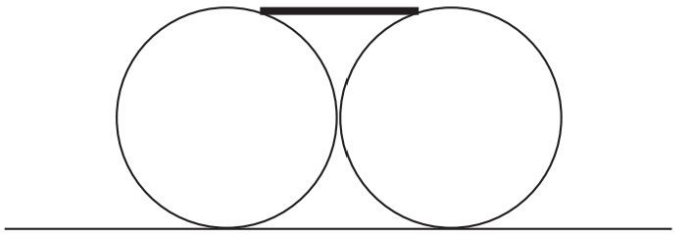
STEP II, 2017, Q9

9 Two identical rough cylinders of radius r and weight W rest, not touching each other but a negligible distance apart, on a horizontal floor. A thin flat rough plank of width $2a$, where $a < r$, and weight kW rests symmetrically and horizontally on the cylinders, with its length parallel to the axes of the cylinders and its faces horizontal. A vertical cross-section is shown in the diagram below.



The coefficient of friction at all four contacts is $\frac{1}{2}$. The system is in equilibrium.

(i) Let F be the frictional force between one cylinder and the floor, and let R be the normal reaction between the plank and one cylinder. Show that

$$R\sin\theta = F(1+\cos\theta)\,,$$

where θ is the acute angle between the plank and the tangent to the cylinder at the point of contact.

Deduce that $2\sin\theta \leqslant 1+\cos\theta$.

(ii) Show that

$$N = \left(1+\frac{2}{k}\right)\left(\frac{1+\cos\theta}{\sin\theta}\right)F\,,$$

where N is the normal reaction between the floor and one cylinder.

Write down the condition that the cylinder does not slip on the floor and show that it is satisfied with no extra restrictions on θ.

(iii) Show that $\sin\theta \leqslant \frac{4}{5}$ and hence that $r \leqslant 5a$.