STEP II, 2017, Q8 MS

Although vectors expressed in general terms are not handled well by the majority of STEP candidates, such questions invariably involve little that is of any great difficulty. If one is sufficiently confident in handling vectors, this question is perhaps the easiest on the paper. The only things involved in this question are the equations of lines in the standard vector form $\mathbf{r} = \mathbf{p} + t\,\mathbf{q}$ and the use of the scalar product for finding angles (in particular the result that, for non-zero vectors $\mathbf{p}$ and $\mathbf{q}$, $\mathbf{p} \bullet \mathbf{q} = 0 \Leftrightarrow \mathbf{p}$ and $\mathbf{q}$ are perpendicular).
Thus, we have the line through A perpendicular to BC is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$ and the line through B perpendicular to CA is $\mathbf{r} = \mathbf{b} + \mu\,\mathbf{v}$, which meet when

$$\mathbf{a} + \lambda\,\mathbf{u} = \mathbf{b} + \mu\,\mathbf{v} \Rightarrow \mathbf{v} = \tfrac{1}{\mu}(\mathbf{a} - \mathbf{b} + \lambda\mathbf{u}).$$

Since $\mathbf{v}$ is perpendicular to CA, $(\mathbf{a}-\mathbf{b}+\lambda\,\mathbf{u})\bullet(\mathbf{a}-\mathbf{c}) = 0$ which leads to a scalar expression for λ and hence a vector expression for $\mathbf{p} = \mathbf{a} + \lambda\,\mathbf{u}$.
Next, $\overrightarrow{CP} = \mathbf{p} - \mathbf{c} = \mathbf{a} - \mathbf{c} + \lambda\mathbf{u}$, and
$\overrightarrow{CP}\bullet\overrightarrow{AB} = (\mathbf{a}-\mathbf{c}+\lambda\mathbf{u})\bullet(\mathbf{b}-\mathbf{a}) = (\mathbf{a}-\mathbf{c})\bullet(\mathbf{b}-\mathbf{a}) + \lambda\mathbf{u}\bullet(\mathbf{b}-\mathbf{a})$.
Now $\mathbf{u}\bullet(\mathbf{b}-\mathbf{c}) = 0$ since $\mathbf{u}$ is perpendicular to $BC \Rightarrow \mathbf{u}\bullet\mathbf{b} = \mathbf{u}\bullet\mathbf{c}$. Substituting this into $\overrightarrow{CP}\bullet\overrightarrow{AB}$ leads very quickly to the required zero for the perpendicularity result required.
$(\mathbf{a}-\mathbf{c})\bullet(\mathbf{b}-\mathbf{a}) + \lambda\mathbf{u}\bullet(\mathbf{c}-\mathbf{a})$.

[**Note:** those readers familiar with the common geometric "centres" of triangles will, no doubt, have spotted that this question is about nothing more than the *orthocentre* of a triangle; that is, the point at which the three altitudes meet. In this question, you are given two altitudes; find their point of intersection, and then show that the line from the third vertex through this point meets the opposite side at right-angles.]

Question 8

Line thro' A perpr. to BC is $\mathbf{r} = \mathbf{a} + \lambda\,\mathbf{u}$	**B1**	
Line thro' B perpr. to CA is $\mathbf{r} = \mathbf{b} + \mu\,\mathbf{v}$	**B1**	
Lines meet when $(\mathbf{r} = \mathbf{p} =)\ \mathbf{a} + \lambda\,\mathbf{u} = \mathbf{b} + \mu\,\mathbf{v}$	**M1**	Equated
$\Rightarrow\ \mathbf{v} = \frac{1}{\mu}(\mathbf{a} - \mathbf{b} + \lambda\,\mathbf{u})$	**A1**	
Since $\mathbf{v}$ is perpr. to CA, $(\mathbf{a} - \mathbf{b} + \lambda\,\mathbf{u})\bullet(\mathbf{a} - \mathbf{c}) = 0$	**M1**	
$\Rightarrow\ (\mathbf{a} - \mathbf{b})\bullet(\mathbf{a} - \mathbf{c}) + \lambda\,\mathbf{u}\bullet(\mathbf{a} - \mathbf{c}) = 0$	**A1**	Correctly multiplied out
$\Rightarrow\ \lambda = \frac{(\mathbf{b} - \mathbf{a})\bullet(\mathbf{a} - \mathbf{c})}{\mathbf{u}\bullet(\mathbf{a} - \mathbf{c})}$	**M1**	Re-arranging for λ
	A1	Correct (any sensible form)
$\Rightarrow\ \mathbf{p} = \mathbf{a} + \left(\frac{(\mathbf{b} - \mathbf{a})\bullet(\mathbf{a} - \mathbf{c})}{\mathbf{u}\bullet(\mathbf{a} - \mathbf{c})}\right)\mathbf{u}$	**A1**	FT their λ (if only **a**, **b**, **c**, **u** involved) **9**
$\overrightarrow{CP} = \mathbf{p} - \mathbf{c} = \mathbf{a} - \mathbf{c} + \lambda\mathbf{u}$	**B1**	FT their λ
Attempt at $\overrightarrow{CP}\bullet\overrightarrow{AB}$	**M1**	
$= (\mathbf{a} - \mathbf{c} + \lambda\mathbf{u})\bullet(\mathbf{b} - \mathbf{a})$	**A1**	Correct to here
$= (\mathbf{a} - \mathbf{c})\bullet(\mathbf{b} - \mathbf{a}) + \lambda\mathbf{u}\bullet(\mathbf{b} - \mathbf{a})$		
Now $\mathbf{u}\bullet(\mathbf{b} - \mathbf{c}) = 0$ since $\mathbf{u}$ perpr. to BC	**M1**	
$\Rightarrow\ \mathbf{u}\bullet\mathbf{b} = \mathbf{u}\bullet\mathbf{c}$	**A1**	
so that $\overrightarrow{CP}\bullet\overrightarrow{AB} = (\mathbf{a} - \mathbf{c})\bullet(\mathbf{b} - \mathbf{a}) + \lambda\mathbf{u}\bullet(\mathbf{c} - \mathbf{a})$	**M1**	Substituted in
$= (\mathbf{a} - \mathbf{c})\bullet(\mathbf{a} - \mathbf{b} + \lambda\mathbf{u})$	**M1 A1**	Factorisation attempt; correct
$= 0$ from boxed line above	**A1 E1**	Statement; justified
$\Rightarrow\ CP$ is perpr. to AB	**E1**	For final, justified statement **11**

Notice that the "value" of   is never actually required

Any candidate who states the result is true because P is the *orthocentre* of ΔABC may be awarded **B2** for actually knowing something about triangle-geometry, but only in addition to any of the first 3 marks earned in the above solution: i.e. a maximum of 5/11 for the second part of the question.