STEP II, 2017, Q7 MS

In this question the difficulty is being able to see that some result is "obviously" true but then having great difficulty in justifying it from particular starting-points: it is not enough to make a true statement (especially when it is given in the question) … one must justify it fully from given, or known, facts and careful deductive reasoning.

Here, in (i), it is known that, for $0 < x < 1$, x is positive and $\ln x$ is negative. Thus $0 > x \ln x > \ln x$ can be deduced by multiplying the first inequality throughout by a negative quantity (remembering that this reverses the direction of inequality signs). This is just $\ln 1 > \ln x^x > \ln x$ and, since the logarithmic function is strictly increasing, $(1 >)\ \mathrm{f}(x) > x$. A more complete argument along similar lines shows that $x < \mathrm{g}(x) < \mathrm{f}(x)$. The final part requires no further justification; since for $x > 1$, $\ln x > 0$ we now have $x < \mathrm{f}(x) < \mathrm{g}(x)$.

For part (ii), it is customary to use logs first and then differentiate implicitly.

In (iii), only an informal understanding regarding the justification of limits is expected, but one still should have a grasp as to how things should be set out. Here, something along the lines of

$$\lim_{x\to 0}\left(\mathrm{f}(x)\right) = \lim_{x\to 0}\left(\mathrm{e}^{x\ln x}\right) = \lim_{x\to 0}\left(\mathrm{e}^{0}\right) = 1$$

and so $\lim_{x\to 0}\left(\mathrm{g}(x)\right) = \lim_{x\to 0}\left(x^{\mathrm{f}(x)}\right) = \lim_{x\to 0}\left(x^{1}\right) = 0$

would be expected.

In (iv), the use of calculus is the most straightforward approach, differentiating $y = \frac{1}{x} + \ln x$ for $x > 0$ and showing that it has a unique minimum turning point at $(1, 1)$. This is then fed in to the derivative of $\mathrm{g}(x)$ – again using the logarithmic form and implicit differentiation – along with a simple observation that squares are necessarily non-negative and this all falls nicely into place. Most of what is required in order to sketch x, $\mathrm{f}(x)$ and $\mathrm{g}(x)$ has already been established and all that is left is to put it together in a sensibly-sized diagram.

(i) For $0 < x < 1$, x is positive and $\ln x$ is negative
so $0 > x \ln x > \ln x$
$\Rightarrow e^0 > e^{x \ln x} > e^{\ln x}$ or $\ln 1 > \ln x^x > \ln x$
$\Rightarrow (1 >)\, f(x) > x$ since ln is a strictly increasing fn. **B1**
Again, since $\ln x < 0$, it follows that
$\ln x < f(x) \ln x < x \ln x$
$\Rightarrow \ln x < \ln\{g(x)\} < \ln\{f(x)\}$ — **M1** Suitably coherent justification
$\Rightarrow \quad x < g(x) < f(x)$ — **A1** **Given Answer** legitimately obtained

For $x > 1$, $\ln x > 0$ and so $x < f(x) < g(x)$ — **B1** No justification required **4**

(ii) $\ln\{f(x)\} = x \ln x$ — **M1** Taking logs and attempting implicit diffn. **Alt.** Writing $y = e^{x \ln x}$ and diffg.

$\frac{1}{f(x)}.f'(x) = x.\frac{1}{x} + 1.\ln x$ i.e. $f'(x) = (1 + \ln x)f(x)$ — **A1**

$f'(x) = 0$ when $1 + \ln x = 0$, $\ln x = -1$, $x = e^{-1}$ — **A1** **3**

(iii) $\lim_{x \to 0} (f(x)) = \lim_{x \to 0} \left(e^{x \ln x}\right) = \lim_{x \to 0} \left(e^0\right) = 1$ — **B1** Suitably justified

$\lim_{x \to 0} (g(x)) = \lim_{x \to 0} \left(x^{f(x)}\right) = \lim_{x \to 0} \left(x^1\right) = 0$ — **B1** May just be stated

Alt. $\lim_{x \to 0} (g(x)) = \lim_{x \to 0} \left(e^{f(x) \ln x}\right) = \lim_{x \to 0} \left(e^{\ln x}\right) = \lim_{x \to 0} (x) = 0$ **2**

(iv) For $y = \frac{1}{x} + \ln x$ $(x > 0)$,

$\frac{dy}{dx} = -\frac{1}{x^2} + \frac{1}{x}$ or $\frac{x-1}{x^2} = 0$ … — **M1** Diffg. and equating to zero

… when $x = 1$ — **A1** From correct derivative

For $x = 1-$, $\frac{dy}{dx} < 0$ and for $x = 1+$, $\frac{dy}{dx} > 0$ — **M1** Method for deciding

(1, 1) is a MINIMUM of $y = \frac{1}{x} + \ln x$ — **A1**

(Since there are no other TPs or discontinuities)
$y \geq 1$ for all $x > 0$ — Conclusion must be made for all 4 marks **4**

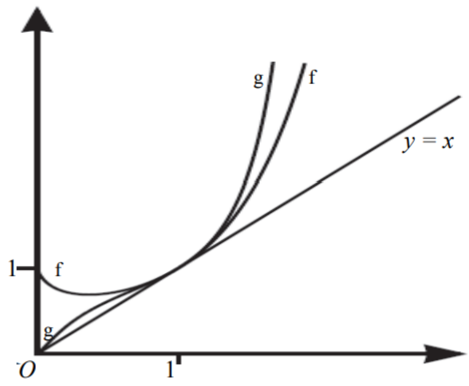
$\ln(g(x)) = f(x) \ln x$ — **M1** Taking logs and attempting implicit diffn.

$\frac{1}{g(x)}.g'(x) = f(x).\frac{1}{x} + \ln x\{f(x)(1 + \ln x)\}$ — **A1** using $f'(x)$ from **(ii)**

$\Rightarrow g'(x) = f(x).g(x)\left\{\frac{1}{x} + \ln x + (\ln x)^2\right\}$

$\geq f(x).g(x)\{1 + (\ln x)^2\}$ — **M1** using previous result of **(iv)**

> 0 since f, g > 0 from **(i)**
and $1 + (\ln x)^2 \geq 1 > 0$ — **A1** **Given Answer** fully justified **4**



B1 One of f, g correct …
B1 Both correct …
… relative to $y = x$

B1 All three passing thro' (1, 1)

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