STEP II, 2017, Q7

7 The functions f and g are defined, for $x > 0$, by

$$\mathrm{f}(x) = x^x\,, \qquad \mathrm{g}(x) = x^{\mathrm{f}(x)}\,.$$

(i) By taking logarithms, or otherwise, show that $\mathrm{f}(x) > x$ for $0 < x < 1$. Show further that $x < \mathrm{g}(x) < \mathrm{f}(x)$ for $0 < x < 1$.

Write down the corresponding results for $x > 1$.

(ii) Find the value of x for which $\mathrm{f}'(x) = 0$.

(iii) Use the result $x \ln x \to 0$ as $x \to 0$ to find $\lim\limits_{x\to 0} \mathrm{f}(x)$, and write down $\lim\limits_{x\to 0} \mathrm{g}(x)$.

(iv) Show that $x^{-1} + \ln x \geqslant 1$ for $x > 0$.

Using this result, or otherwise, show that $\mathrm{g}'(x) > 0$.

Sketch the graphs, for $x > 0$, of $y = x$, $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ on the same axes.