STEP II, 2017, Q6 MS

This question is about two different types of proof – induction and direct manipulation. Both of which in isolation are generally well understood, but it is very easy to get lost in the algebra, especially with the added complication of inequalities.

Part (i) explicitly required induction, so there is a standard procedure to follow – check it works when $n = 1$, assume it works when $n = k$ and show that this leads to it being true when $n = k + 1$. Only the last part causes any issue. There is some fairly subtle logic: using the $n = k$ assumption it can be shown that

$$S_{k+1} \leq 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}}$$

However, what is really needed is:

$$S_{k+1} \leq 2\sqrt{k+1} - 1$$

One way in which this can be established (technically a sufficient, but not necessary condition) is if it can be shown that

$$2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} - 1$$

A bit of tidying and rearranging shows that this is equivalent to

$$4k^2 + 4k \leq 4k^2 + 4k + 1$$

which is "obviously" true. You might worry a little about the fact that the equality is never satisfied, but showing that the strict inequality is true is sufficient.

The first part of part (ii) is also just about squaring up and showing that the statement is equivalent to an "obviously" true statement (in this case that $16k^3 + 24k^2 + 9k + 1 > 16k^3 + 24k^2 + 9k$). No induction required! But why are we being asked to do this....?

In the final part we first need to come up with a conjecture. A reasonable place to start is to try $n = 1$, so that

$$1 \geq 2.5 - C$$

For this to work we need that $C \geq 1.5$ so $C = 1.5$ is the smallest value that works for S_1. However will this work for all subsequent S_n too? It turns out that it does, but that requires proving the conjecture

$$S_n \geq 2\sqrt{n} + \frac{1}{\sqrt{n}} - 1.5$$

This requires induction, following a very similar argument to part (i). One line of the algebra in the proof requires $(4k+1)\sqrt{k+1} \geq (4k+3)\sqrt{k}$, which can be done using the initially unimportant fact at the beginning of part (ii) – always look out for making links between the different parts of questions!

(i)		
When $n = 1$ $$S_1 = 1 \le 2\sqrt{1} - 1$$	**B1**	Clear verification.
Assume that the statement is true when $n = k$: $$S_k \le 2\sqrt{k} - 1$$	**B1**	Must be clear that this is assumed.
Then $$S_{k+1} = S_k + \frac{1}{\sqrt{k+1}}$$	**M1**	Linking S_{k+1} and S_k
$$\le 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}}$$	**M1**	Using assumed result
Sufficient to prove: $$2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} \le 2\sqrt{k+1} - 1$$	**M1**	
i.e. $2\sqrt{k(k+1)} + 1 \le 2(k+1)$	**A1**	Multiplying by $\sqrt{k+1}$ or putting over a common denominator
i.e. $2\sqrt{k(k+1)} \le 2k + 1$		
i.e. $4k^2 + 4k \le 4k^2 + 4k + 1$	**A1**	
Which is clearly true. Therefore by induction the statement is true for all $n \ge 1$.	**B1**	Clear conclusion showing logic of induction.
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(ii)		
Required to prove: $$(4k+1)^2(k+1) > (4k+3)^2 k$$	**M2**	Squaring given inequality
i.e. $16k^3 + 24k^2 + 9k + 1 > 16k^3 + 24k^2 + 9k$ which is clearly true.	**A1**	
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When $n = 1$: $$S_1 = 1 \ge 2 + \frac{1}{2} - c$$	**M1**	
So we need $c \ge \frac{3}{2}$	**A1**	
Prove $c = \frac{3}{2}$ works using induction	**M1**	
Assume holds when $n = k$: $$S_k \ge 2\sqrt{k} + \frac{1}{2\sqrt{k}} - \frac{3}{2}$$	**M1**	Allow a general c.
Then $$S_{k+1} = S_k + \frac{1}{\sqrt{k+1}} \ge 2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} - c$$	**M1**	
Sufficient to prove: $$2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} - c \ge 2\sqrt{k+1} + \frac{1}{2\sqrt{k+1}} - c$$	**A1**	
i.e. $4k\sqrt{k+1} + \sqrt{k+1} + 2\sqrt{k} \ge 4\sqrt{k}(k+1) + \sqrt{k}$	**A1A1**	
Which simplifies to the previously proved inequality. No further restrictions on c, so the minimum value is $c = \frac{3}{2}$	**B1**	
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