STEP II, 2017, Q5 MS

In many ways, much of this question is also relatively routine. Find $\frac{dy}{dx}$ for the gradient of the tangent; find its negative reciprocal for the gradient of the normal and then any one of a number of formulae for the equation of a line. At some stage, you will need to replace t by p for the normal at P and then replace x and y in this equation by an^2 and $2an$ for another point on the curve. Solving for n in terms of p – noting that the factor $(n - p)$ must be involved somewhere, since $n = p$ must be one solution to whatever equation arises as the line is already known to meet the curve at P – should then yield the given answer.
In (ii), employing the distance formula $PN^2 = (x_P - x_N)^2 + (y_P - y_N)^2$ is clearly the way forwards, as is replacing n by $-\left(p + \frac{2}{p}\right)$ at some stage of the proceedings. The rest is just careful algebra. Differentiating the given expression for PN^2 with respect to p is routine enough, in principle, and it is then only required to justify that the (only) values of p that arise will give minimum points. One could use the *first-derivative test* (looking for a change of sign), the *second-derivative test* (examining its sign) or argue from the shape of the curve $(y =)\ \frac{16a^2}{p^4}(p^2 + 1)^3$ ... which is symmetric in the y-axis, asymptotic to the y-axis for small values of p, and can be arbitrarily large as $|p| \to \infty$; thus, any turning-points must be minima.
For part (iii), one starts by noting that PQ and NQ are perpendicular (since $\angle PQN = 90°$, by "Angle in a semi-circle"). Then, setting the products of their gradients, $\frac{2}{p+q}$ and $\frac{2}{n+q}$, equal to – 1, replacing n by $-\left(p + \frac{2}{p}\right)$ once again, and using $p^2 = 2$, takes you almost the whole way there: $q^2 = \frac{2q}{p}$. From this point, we have $q = 0$ or $q = \frac{2}{p} = \pm\sqrt{2}$. Finally, these final two cases should be eliminated by noting that they give $q = p$, i.e. $Q = P$, which is not the case as they are being taken to be distinct points.

(i) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$ — **M1** Finding gradt. of tgt. (or by implicit diffn.)

$\Rightarrow$ Grad. nml. at P is $-p$ — **A1**

$\Rightarrow$ Eqn. nml. to C at P is $x - 2ap = -p(x - ap^2)$ — **B1** **FT** any form, e.g. $y = -px + ap(2 + p^2)$

Nml. meets C again when $x = an^2,\ y = 2an$ — **M1** Substd. into nml. eqn.

$\Rightarrow\ 2an = -pan^2 + ap(2 + p^2)$

$\Rightarrow\ 0 = pn^2 + 2n - p(2 + p^2)$

$\Rightarrow\ 0 = (n - p)(pn + [2 + p^2])$ — **M1** Solving attempt

Since $n = p$ at P, it follows that $n = -\frac{2 + p^2}{p}$ at N

i.e. $n = -\left(p + \frac{2}{p}\right)$ — **A1** **Given Answer** legitimately obtained **6**

(ii) Distance $P(ap^2, 2ap)$ to $N(an^2, 2an)$ is given by

$PN^2 = \left[a(p^2 - n^2)\right]^2 + \left[2a(p - n)\right]^2$ — **M1**

$= a^2(p - n)^2\left\{(p + n)^2 + 4\right\}$

$= a^2\left(2p + \frac{2}{p}\right)^2\left\{\left(\frac{-2}{p}\right)^2 + 4\right\}$ — **M1** Substituting for n

$= 16a^2\left(\frac{p^2 + 1}{p}\right)^2\left\{\frac{1 + p^2}{p}\right\} = 16a^2\frac{(p^2 + 1)^3}{p^4}$ — **A1** **Given Answer** legitimately obtained **3**

$\frac{d(PN^2)}{dp} = 16a^2\frac{d(p^2 + 3 + 3p^{-2} + p^{-4})}{dp}$ — **M1** Differentiation directly, or by the Quotient Rule

$= 16a^2(2p - 6p^{-3} - 4p^{-5})$ — **A1** Correct, unsimplified

$= 32a^2\frac{p^6 - 3p^2 - 2}{p^5}$

$= \frac{32a^2}{p^5}(p^2 + 1)^2\left[p^2 - 2\right]$

Note that $\frac{d(PN^2)}{dp} = 16a^2\left\{\frac{p^4.3(p^2 + 1)^2.2p - (p^2 + 1)^3.4p^3}{p^8}\right\}$

$= \frac{32a^2}{p^8}.p^3(p^2 + 1)^2\left[3p^2 - 2(p^2 + 1)\right]$ by the Quotient Rule

$\frac{d(PN^2)}{dp} = 0$ only when $p^2 = 2$ — **A1** **Given Answer** fully shown

Justification that it is a minimum — **E1**

(either by examining the sign of $\frac{d(PN^2)}{dp}$

or by explaining that PN^2 cannot be maximised **4**

(iii) Grad. PQ is $\dfrac{2}{p+q}$ **B1**

Grad. NQ is $\dfrac{2}{n+q}$ or $\dfrac{2}{q-p-\frac{2}{p}}$ **B1**

Since $\angle PQN = 90°$ (by "$\angle$ in a semi-circle"; i.e. *Thales Theorem*)

$$\frac{2}{p+q} \times \frac{2}{q-p-\frac{2}{p}} = -1$$ **M1**

$$\Rightarrow \; 4 = (p+q)\left(p-q+\frac{2}{p}\right) = p^2 - q^2 + 2 + \frac{2q}{p}$$

$$\Rightarrow \; 2 = p^2 - q^2 + \frac{2q}{p}$$ **A1** **Given Answer** legitimately obtained **4**

PN minimised when $p^2 = 2 \;\Rightarrow\; q^2 = \dfrac{2q}{p}$ **M1** Substituted into given expression

$\Rightarrow \; q = 0$ or $q = \dfrac{2}{p} = \pm\sqrt{2}$ **A1**

But $q = \pm\sqrt{2} \Rightarrow q = p$ (which is not the case) **E1** Other cases must be ruled out

Special Case: 1/3 for substg. $q = 0$ and verifying that $p^2 = 2$ **3**