# STEP II, 2017, Q4 MS

This is an interesting question and very straightforward in some respects. To begin with, you are told in part (i) that f($x$) = 1. At this point, it would be wise to write down what the *Schwarz inequality* gives in this case:

$$\left(\int_a^b g(x)\,dx\right)^2 \le \left(\int_a^b 1\,dx\right)\left(\int_a^b [g(x)]^2\,dx\right);\ \text{i.e.}\ \left(\int_a^b g(x)\,dx\right)^2 \le (b-a)\left(\int_a^b [g(x)]^2\,dx\right).$$

A few moments of careful thought (inspecting the given answer) should make you realise that $a = 0$, $b = t$ give the terms $(b - a) = (t - 0)$ *and* $(e^t - e^0)$ when g($x$) = $e^x$. Following it through from there is relatively routine, provided one spots the *difference-of-two-squares factorisation* and that we can divide throughout by $(e^t + 1)$, which is guaranteed to be positive (an important consideration when dealing with inequalities).

In (ii), it is (again) best to start by seeing how things appear when you have used the given information that f($x$) = $x$ and, by clear implication, $a = 0$ and $b = 1$:

$$\left(\int_0^1 x\,g(x)\,dx\right)^2 \le \left(\int_0^1 x^2\,dx\right)\left(\int_0^1 [g(x)]^2\,dx\right);\ \text{i.e.}\ \left(\int_0^1 x\,g(x)\,dx\right)^2 \le \tfrac{1}{3}\left(\int_0^1 [g(x)]^2\,dx\right).$$

The $e^{-\frac{1}{4}}$ in the given answer, along with the fact that $\left(e^{-\frac{1}{4}x^2}\right)^2 = e^{-\frac{1}{2}x^2}$ should point the way towards choosing g($x$) = $e^{-\frac{1}{4}x^2}$. Following this through carefully again yields the required result.

The result in part (iii) clearly requires the use of the Schwarz inequality twice, once each for the right- and left-halves of the given result. Setting $f(x) = 1$, g($x$) = $\sqrt{\sin x}$, $a = 0$ and $b = \frac{1}{2}\pi$ leads to $\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,dx \le \sqrt{\frac{\pi}{2}}$. However, the left-hand half of the result does require a bit more thought and, preferably, familiarity with the integration of trig. functions where powers of sin $x$ (in this case) appear along with its derivative, cos $x$. The real clue is that, for the $\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,dx$ to appear on the other side of the inequality to the one found using the obvious candidates that led to the right-hand half of the result, $\sqrt{\sin x}$ must now be the result of the squaring process. It is then experience (or insight) that suggests setting f($x$) = $\cos x$, g($x$) = $\sqrt[4]{\sin x}$, $a = 0$ and $b = \frac{1}{2}\pi$. You will then find that the LHS of the *Schwarz inequality* requires the integration of a function of sin $x$ multiplied by its derivative, cos $x$; and this (as in Q1) can be done by "recognition" or substitution. (You will also need to be able to integrate $\cos^2 x$, which calls upon the use of the standard *double-angle formula* for cosine.)

**(i)** Setting $\mathrm{f}(x) = 1$ in **(*)** gives

$$\left(\int_a^b \mathrm{g}(x)\,\mathrm{d}x\right)^2 \le \left(\int_a^b 1\,\mathrm{d}x\right)\left(\int_a^b [\mathrm{g}(x)]^2\,\mathrm{d}x\right)$$

**B1** Clearly stated

Let $\mathrm{g}(x) = \mathrm{e}^x$ : $\left(\int_a^b \mathrm{e}^x\,\mathrm{d}x\right)^2 \le (b-a)\left(\int_a^b \mathrm{e}^{2x}\,\mathrm{d}x\right)$

**M1**

$$\Rightarrow \left(\mathrm{e}^b - \mathrm{e}^a\right)^2 \le (b-a).\tfrac{1}{2}\left(\mathrm{e}^{2b} - \mathrm{e}^{2a}\right)$$

$$\Rightarrow \left(\mathrm{e}^b - \mathrm{e}^a\right)^2 \le (b-a).\tfrac{1}{2}\left(\mathrm{e}^b - \mathrm{e}^a\right)\left(\mathrm{e}^b + \mathrm{e}^a\right)$$

$$\Rightarrow \mathrm{e}^b - \mathrm{e}^a \le \tfrac{1}{2}(b-a)\left(\mathrm{e}^b + \mathrm{e}^a\right)$$

**A1**

Choosing $a = 0$ and $b = t$ gives **M1**

$$\mathrm{e}^t - 1 \le \tfrac{1}{2}t\left(\mathrm{e}^t + 1\right) \Rightarrow \frac{\mathrm{e}^t - 1}{\mathrm{e}^t + 1} \le \tfrac{1}{2}t$$

**A1** **Given Answer** legitimately obtained **5**

**(ii)** Setting $\mathrm{f}(x) = x$, $a = 0$ and $b = 1$ in **(*)** gives

$$\left(\int_0^1 x\,\mathrm{g}(x)\,\mathrm{d}x\right)^2 \le \left(\int_0^1 x^2\,\mathrm{d}x\right)\left(\int_0^1 [\mathrm{g}(x)]^2\,\mathrm{d}x\right)$$

**B1** Clearly stated

Choosing $\mathrm{g}(x) = \mathrm{e}^{-\frac{1}{4}x^2}$ gives **M1**

$$\left(\int_0^1 x\mathrm{e}^{-\frac{1}{4}x^2}\,\mathrm{d}x\right)^2 \le \tfrac{1}{3}\left(1^3 - 0^3\right)\left(\int_0^1 \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x\right)$$

$$\left(\left[-2\mathrm{e}^{-\frac{1}{4}x^2}\right]_0^1\right)^2 \le \tfrac{1}{3}\left(\int_0^1 \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x\right)$$

**A1 A1** LHS, RHS correct

$$\Rightarrow \int_0^1 \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x \ge 3\left(-2\left[-\mathrm{e}^{-\frac{1}{4}} + 1\right]\right)^2$$

i.e. $\int_0^1 \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x \ge 12\left(1 - \mathrm{e}^{-\frac{1}{4}}\right)^2$

**A1** **Given Answer** legitimately obtained **5**

**(iii)** With $\mathrm{f}(x) = 1$, $\mathrm{g}(x) = \sqrt{\sin x}$, $a = 0$, $b = \frac{1}{2}\pi$,

**M1** Correct choice for f, g (or v.v.)

**(*)** becomes

**M1** Any sensible f, g used in **(*)**

$$\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,\mathrm{d}x\right)^2 \le \tfrac{1}{2}\pi\left(\int_0^{\frac{1}{2}\pi} \sin x\,\mathrm{d}x\right)$$

**A1**

RHS is $\tfrac{1}{2}\pi\,\left[-\cos x\right]_0^1 = \tfrac{1}{2}\pi$

(and since LHS is positive) we have $\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,\mathrm{d}x \le \sqrt{\dfrac{\pi}{2}}$

**A1** RH half of **Given** inequality obtained from fully correct working **4**

With $\mathrm{f}(x) = \cos x$, $\mathrm{g}(x) = \sqrt[4]{\sin x}$, $a = 0$, $b = \frac{1}{2}\pi$, **M1** Correct choice for f, g (or v.v.)

**(*)** gives

$$\left(\int_0^{\frac{1}{2}\pi} \cos x.(\sin x)^{\frac{1}{4}}\, dx\right)^2 \le \left(\int_0^{\frac{1}{2}\pi} \cos^2 x\, dx\right)\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\, dx\right)$$ **A1**

$$\text{LHS} = \left(\left[\tfrac{4}{5}(\sin x)^{\frac{5}{4}}\right]_0^{\frac{1}{2}\pi}\right)^2 = \tfrac{16}{25}$$ **M1 A1** By recognition/substitution integration

$$\text{and} \quad \int_0^{\frac{1}{2}\pi} \cos^2 x\, dx = \int_0^{\frac{1}{2}\pi} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right) dx$$ **M1**

$$= \left(\left[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\frac{1}{2}\pi}\right)^2 = \tfrac{1}{4}\pi$$ **A1**

Giving the required LH half of the **Given** inequality:

$$\tfrac{16}{25} \le \tfrac{1}{4}\pi\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\, dx\right) \quad \text{i.e.} \quad \int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\, dx \ge \frac{64}{25\pi}$$

**6**

Withhold the last A mark if final result is not arrived at