STEP II, 2017, Q4

4 The Schwarz inequality is

$$\left(\int_a^b \mathrm{f}(x)\,\mathrm{g}(x)\,\mathrm{d}x\right)^2 \leqslant \left(\int_a^b (\mathrm{f}(x))^2\mathrm{d}x\right)\left(\int_a^b (\mathrm{g}(x))^2\mathrm{d}x\right). \qquad (*)$$

(i) By setting $\mathrm{f}(x) = 1$ in $(*)$, and choosing $\mathrm{g}(x)$, a and b suitably, show that for $t > 0$,

$$\frac{\mathrm{e}^t - 1}{\mathrm{e}^t + 1} \leqslant \frac{t}{2}.$$

(ii) By setting $\mathrm{f}(x) = x$ in $(*)$, and choosing $\mathrm{g}(x)$ suitably, show that

$$\int_0^1 \mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x \geqslant 12\big(1 - \mathrm{e}^{-\frac{1}{4}}\big)^2.$$

(iii) Use $(*)$ to show that

$$\frac{64}{25\pi} \leqslant \int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,\mathrm{d}x \leqslant \sqrt{\frac{\pi}{2}}.$$