STEP II, 2017, Q3 MS

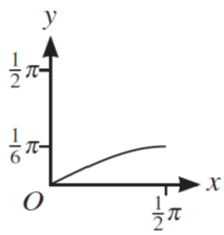
Whilst this question might appear somewhat daunting on first reading, it involves little more than an understanding of the sine curve and the key results that relate to "angles in all quadrants".

To begin with, "$\sin y = \sin x \Rightarrow y = x$" is the kind of response expected from those candidates who have failed to understand that there are many functions (even the simple ones covered at A-level) that don't map "one-to-one". Though the general formula "$\sin y = \sin x \Rightarrow y = n\pi + (-1)^n x$" might be unfamiliar, it only says that if $\sin y = \sin x$ (imagining x to be positive and acute) then y could be any "first quadrant" equivalent of x ... an even multiple of π plus x ... or a "second quadrant" supplement of x ... an odd multiple of π minus x. (Apart from that, y's corresponding to non-acute x's arise from application of the same principles, with "quadrants" taking care of themselves due to the symmetries of the sine curve.)

In (i), it is then found that the general formula (or its equivalent in two bits) gives three graphs that are of interest here: $n = -1$ ($y = -\pi - x$), $n = 0$ ($y = x$) and $n = 1$ ($y = \pi - x$), all of which give straight-line segments in the interval required.

In (ii), there is rather less thinking to be done – at least to begin with – since one is only required to differentiate twice and do some tidying up. This calculus can be done implicitly or directly after rearranging into an arcsine form. That is not to say that it is easy calculus, since the *Product*, *Quotient* and *Chain* rules can all play a part in the processes that follow.

In sketching the graph, one should start simple and work up. Initially, $\frac{dy}{dx} = \frac{1}{2}$ at (0, 0), with the curve increasing to a maximum at $(\frac{\pi}{2}, \frac{\pi}{6})$, since $\frac{d^2y}{dx^2} < 0$. This gives



Thereafter, for whatever "other" bits there are to the curve, these follow from symmetries, applied to the above portion: namely, reflection symmetry in $x = \frac{\pi}{2}$; rotational symmetry about *O*; and reflection symmetry in $y = \pm\frac{\pi}{2}$.

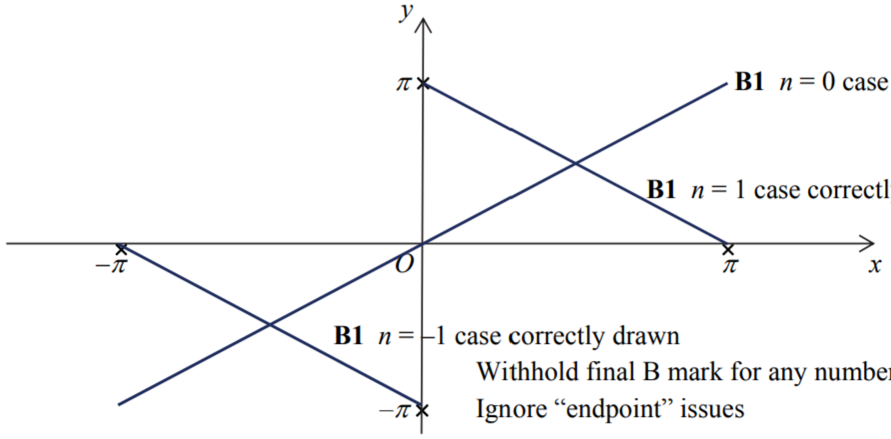
Part (iii)'s graph follows by applying the result $\cos y \equiv \sin(\frac{\pi}{2} - y)$.

(i) $\sin y = \sin x \Rightarrow y = n\pi + (-1)^n x$

$n = -1$: $y = -\pi - x$ **B1**

$n = 0$: $y = x$ **B1**

$n = 1$: $y = \pi - x$ **B1** Withhold final B mark for any number of extra eqns.



B1 $n = 0$ case correctly drawn

B1 $n = 1$ case correctly drawn

B1 $n = -1$ case correctly drawn

Withhold final B mark for any number of extra lines

Ignore "endpoint" issues

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(ii) $\sin y = \frac{1}{2}\sin x \Rightarrow \cos y \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\cos x$ **M1** Implicit diffn. attempt (or equivalent)

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{2\cos y}$ **A1** Correct

$= \frac{\cos x}{2\sqrt{1 - \frac{1}{4}\sin^2 x}}$ or $\frac{\cos x}{\sqrt{4 - \sin^2 x}}$ **A1** Correct and in terms of x only **3**

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{(4 - \sin^2 x)^{\frac{1}{2}} . - \sin x - \cos x . \frac{1}{2}(4 - \sin^2 x)^{-\frac{1}{2}} . -2\sin x \cos x}{4 - \sin^2 x}$$

M1 For use of the *Quotient Rule* (or equivalent)

M1 For use of the *Chain Rule* for d/dx(denominator)

A1

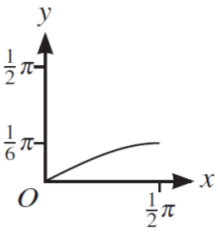
$$= \frac{-\sin x(4 - \sin^2 x) + \cos^2 x . \sin x}{(4 - \sin^2 x)^{\frac{3}{2}}}$$

M1 Method for getting correct denominator

$$= \frac{\sin x\{\cos^2 x - 4 + \sin^2 x\}}{(4 - \sin^2 x)^{\frac{3}{2}}}$$

$$= \frac{-3\sin x}{(4 - \sin^2 x)^{\frac{3}{2}}}$$

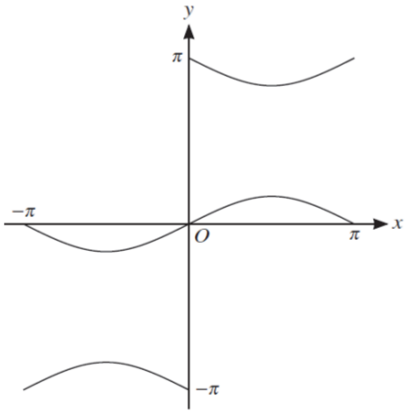
A1 **Given Answer** correctly obtained from $c^2 + s^2 = 1$ **5**



Initially, $\frac{dy}{dx} = \frac{1}{2}$ at (0, 0) increasing to a maximum

at $\left(\frac{\pi}{2}, \frac{\pi}{6}\right)$ since $\frac{d^2y}{dx^2} < 0$

B1 (Gradient and coordinate details unimportant unless graphs look silly as a result)



B1 Reflection symmetry in $x = \frac{\pi}{2}$

B1 Rotational symmetry about O

B1 Reflection symmetry in $y = \pm \frac{\pi}{2}$

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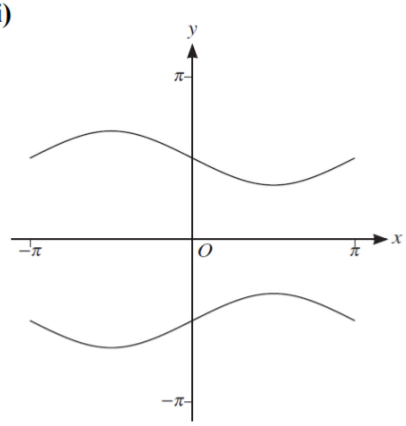
(iii)



B1 RHS correct

B1 LHS correct

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