STEP II, 2017, Q2 MS

The opening part of the question is essentially identical to the process of *composition of functions*. Moreover, if subscripts are likely to prove confusing, then begin with a statement such as "Let $x_n = X$." Thus $x_{n+1} = \dfrac{aX-1}{X+b}$ and $x_{n+2} = \dfrac{a\left(\dfrac{aX-1}{X+b}\right)-1}{\left(\dfrac{aX-1}{X+b}\right)+b}$, which simply needs tidying up. This yields $x_{n+2} = \dfrac{(a^2-1)X-(a+b)}{(a+b)X+(b^2-1)}$. The "**Note**" in (i) reminds the reader that the period of a periodic sequence is the length of the *smallest* cycle of repetition; thus, we require $x_{n+2} = x_n$ but **not** $x_{n+1} = x_n$. A moment's thought should reveal it to be obvious that a constant sequence should still yield part of the same algebra which gives $x_{n+2} = x_n$ and it is worth exploring this first:

if $x_{n+1} = x_n$ then $aX - 1 = X^2 + bX \Rightarrow 0 = X^2 - (a-b)X + 1$.

One might be tempted to try to solve this (using the *Quadratic formula*, for instance), but there is really nothing to be gained by so doing, since the constant sequence is of no interest to us, only the conditions that give it (which we need to have in mind later on). Proceeding to explore $x_{n+2} = x_n$ gives us, upon collecting up, $0 = (a+b)\{X^2 - (a-b)X + 1\}$. Fortunately, the factor $(a+b)$ is readily apparent, but the quadratic factor should have been anticipated also, for the reasons outlined above. Thus, $a + b = 0$ is a necessary condition for a period 2 sequence. (There is no requirement to explore the issue of sufficiency, which could be done by setting $b = -a$ in the initial expression for x_{n+1} and then following it through to see what happens.)

Finally, we are asked to see what happens when $x_{n+4} = x_n$, and this can be done the long way round by finding $x_{n+3} = \dfrac{(a^3-2a-b)X-(a^2+ab+b^2-1)}{(a^2+ab+b^2-1)X-(a+2b+b^3)}$ and then $x_{n+4} = \dfrac{ax_{n+3}-1}{x_{n+3}+b}$, but there is at least one very obvious shortcut to what is starting to look like some complicated algebra: namely, considering the "two-step" result already found and repeating that, going from x_n to x_{n+4} via x_{n+2}. It also helps if one realises that whatever algebraic expression appears, we know that it must have the previously found factors of $(a+b)$ and $\{X^2 - (a-b)X + 1\}$ within it.

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Let $x_n = X$. Then $x_{n+1} = \dfrac{aX-1}{X+b}$ and $x_{n+2} = \dfrac{a\left(\dfrac{aX-1}{X+b}\right)-1}{\left(\dfrac{aX-1}{X+b}\right)+b}$ **M1 A1** Correct, unsimplified

i.e. $x_{n+2} = \dfrac{(a^2-1)X-(a+b)}{(a+b)X+(b^2-1)}$ **M1** Attempt to remove "fractions within fractions"

A1 Correct, simplified **4**

(i) If $x_{n+1} = x_n$ then $aX - 1 = X^2 + bX$ **M1**

$\Rightarrow\ 0 = X^2 - (a-b)X + 1$ **A1**

If $x_{n+2} = x_n$ then

$(a^2-1)X - (a+b) = (a+b)X^2 + (b^2-1)X$ **M1**

$\Rightarrow\ 0 = (a+b)\{X^2 - (a-b)X + 1\}$ **M1 A1** Factorisation

and so, for $x_{n+2} = x_n$ but $x_{n+1} \neq x_n$

we must have $a + b = 0$ **A1** **Given Answer** fully justified & clearly stated

(No marks for setting $b = -a$, for instance, and showing sufficiency)

For "comparing coefficients" approach (must be all 3 terms) max. 3/4 **6**

(ii) $x_{n+4} = \dfrac{(a^2-1)x_{n+2}-(a+b)}{(a+b)x_{n+2}+(b^2-1)}$ **M1** Use of the two-step result from earlier

$= \dfrac{(a^2-1)\left[\dfrac{(a^2-1)X-(a+b)}{(a+b)X+(b^2-1)}\right]-(a+b)}{(a+b)\left[\dfrac{(a^2-1)X-(a+b)}{(a+b)X+(b^2-1)}\right]+(b^2-1)}$ **A1** Correct, unsimplified, in terms of X

If $x_{n+4} = x_n$ then **M1** Equating

$(a^2-1)^2X - (a+b)(a^2-1) - (a+b)^2X - (a+b)(b^2-1)$ **A1** LHS correct

$= (a+b)(a^2-1)X^2 - (a+b)^2X + (a+b)(b^2-1)X^2 + (b^2-1)^2X$ **A1** RHS correct

$\Rightarrow\ 0 = (a+b)(a^2+b^2-2)X^2 - [(a^2-1)^2 - (b^2-1)^2]X + (a+b)(a^2+b^2-2)$

M1 Good attempt to simplify

$\Rightarrow\ 0 = (a+b)(a^2+b^2-2)\{X^2 - (a-b)X + 1\}$ **M1** Factorisation attempt

A1 A1 Partial; complete

and the sequence has period 4 if and only if

$a^2 + b^2 = 2,\ a + b \neq 0,\ X^2 - (a-b)X + 1 \neq 0$ **B1 CAO** Correct final statement

[Ignore any discussion or confusion regarding issues of necessity and sufficiency]

NB Some candidates may use the one-step result repeatedly and get to x_{n+4} via x_{n+3}:

$x_{n+3} = \dfrac{(a^3-2a-b)X-(a^2+ab+b^2-1)}{(a^2++ab+b^2-1)X-(a+2b+b^3)}$ and $x_{n+4} = \dfrac{ax_{n+3}-1}{x_{n+3}+b}$ starts the process; then as above.

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ALT. Consider the two-step sequence $\{\ldots, x_n, x_{n+2}, x_{n+4}, \ldots\}$ given by (assuming $a + b \neq 0$)

$$x_{n+2} = \frac{\left(\frac{a^2-1}{a+b}\right)X - 1}{X + \left(\frac{b^2-1}{a+b}\right)} \equiv \frac{AX-1}{X+B},$$ which is clearly of exactly the same form as before.

Then $x_{n+4} = x_n$ if and only if $a + b \neq 0$, $X^2 - (a-b)X + 1 \neq 0$ (from $x_{n+4} \neq x_{n+2}$ and $x_{n+4} \neq x_n$ as before), together with the condition $A + B = 0$ (also from previous work);

i.e. $\frac{a^2-1}{a+b} + \frac{b^2-1}{a+b} = 0$, which is equivalent to $a^2 + b^2 - 2 = 0$ since $a + b \neq 0$.

Note that it is not necessary to consider $x_{n+4} \neq x_{n+3}$ since if $x_{n+4} = x_{n+3} = X$ then the sequence would be constant.

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