STEP II, 2017, Q1 MS

The opening "Note" clearly flags a result which will prove important in this question; this is a 'standard' result, but one that can slip by unnoticed in single-maths A-levels; it is, therefore, worth learning as an "extra", if necessary.

In (i), it should be obvious that the process of *integration by parts* is to be used and that the initial hint indicates that the "first part" must be the "arctan x" term, despite appearing as the second term of the product to be integrated. This will lead directly to the given result, following which the substitution of $n = 0$ leads to the integral $\int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x$. This can be done by "recognition" (or *reverse-Chain Rule* integration) or by substitution. In the first instance, one would note that $\int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,\mathrm{d}x$, where the numerator is precisely the derivative of the denominator – the standard "log. integral" form; in the second instance, the substitution $u = 1 + x^2$ will also work, though it might take just a few lines more working.

In (ii), one needs only to use the result given in (i), this time replacing n by $(n + 2)$ to find an expression for $(n + 3)I_{n+2}$; and then adding to it the given expression for $(n + 1)I_n$. This leads to a result in which two integrals must be added to get, when simplified, $\int_0^1 \frac{x^{n+1}\left(1+x^2\right)}{1+x^2}\,\mathrm{d}x$, which should need no further comment. Setting $n = 0$ and then $n = 2$ in this result then yields a numerical answer for I_4, since I_0 has already been calculated.

In (iii), no matter how demanding the process of mathematical induction appears to be, it is very formulaic in some respects and there are always some marks to be had. To begin with, there is always the "baseline" case of (usually) $n = 1$. In this case, one must set $n = 1$ in the proposed formula and check it against the value of I_4 already known. This reveals the value of A to be $\frac{1}{4}\pi - \frac{1}{2}\ln 2$. However, since it is constant, it remains fixed during all of the remaining work and one can most easily progress through the rest of the inductive proof by simply continuing to use the letter A.

A clearly stated induction hypothesis is enormously helpful (usually replacing the n by another dummy index, k say) rather than vague statements such as "assume the result is true for $n = k$" or meaningless statements such as "assume $n = k$". We thus proceed assuming $(4k + 1)\, I_{4k+1} = A - \frac{1}{2}\sum_{r=1}^{2k}(-1)^r \frac{1}{r}$.

The remainder of the proof relies more on carefulness than inspiration, especially as the process relies on exactly the same techniques as were used in part (ii), using k and $(k + 1)$ in turn in the given result.

(i) $I_n = \int_0^1 \arctan x \,.\, x^n \, dx$ — **M1** Use of intgrn. by parts (parts correct way round)

$= \left[\arctan x \,.\, \frac{x^{n+1}}{n+1}\right]_0^1 - \int_0^1 \frac{1}{1+x^2} \cdot \frac{x^{n+1}}{n+1} dx$ — **A1** Correct to here

$= \left(\frac{\pi}{4} \cdot \frac{1}{n+1} - 0\right) - \frac{1}{n+1} \int_0^1 \frac{x^{n+1}}{1+x^2} dx$

$\Rightarrow \; (n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2} dx$ — **A1** Given Answer legitimately established **3**

Setting $n = 0$, $I_0 = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} dx$ — **M1** Attempt to solve this using recognition/ substitution

$= \frac{\pi}{4} - \left[\tfrac{1}{2}\ln\left(1+x^2\right)\right]$ — **M1** Log integral involved

$= \frac{\pi}{4} - \tfrac{1}{2}\ln 2$ — **A1** CAO **3**

(ii) $n \to n+2$ in given result:

$(n+3)I_{n+2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2} dx$ — **B1** Noted or used somewhere

$(n+3)I_{n+2} + (n+1)I_n = \frac{\pi}{2} - \int_0^1 \frac{x^{n+1}\left(1+x^2\right)}{1+x^2} dx$ — **M1** Adding and cancelling ready to integrate

$= \frac{\pi}{2} - \frac{1}{n+2}$ — **A1** CAO **3**

Setting $n = 0$ and then $n = 2$ in this result (or equivalent involving integrals):

$3I_2 + I_0 = \frac{\pi}{2} - \frac{1}{2}$ and $5I_4 + 3I_2 = \frac{\pi}{2} - \frac{1}{4}$ — **M1**

Eliminating I_2 and using value for I_0 to find I_4 — **M1** By subtracting, or equivalent

$I_4 = \tfrac{1}{20}\left(1 + \pi - 2\ln 2\right)$ — **A1** **FT** from their I_0 value **3**

(iii) For $n = 1$, $5I_4 = A - \tfrac{1}{2}\left(-1 + \tfrac{1}{2}\right) = A + \tfrac{1}{4}$

$= \tfrac{1}{4} + \tfrac{1}{4}\pi - \tfrac{1}{2}\ln 2$ — **M1** Comparing formula with found I_4 value

and the result is true for $n = 1$ provided

$A = \tfrac{1}{4}\pi - \tfrac{1}{2}\ln 2$ — **A1** **FT** from their I_4 value **2**

Assuming $(4k+1)I_{4k+1} = A - \frac{1}{2}\sum_{r=1}^{2k}(-1)^r \frac{1}{r}$ **M1** For a clearly stated induction hypothesis

(or a fully explained "if … then …" at end)

$(4k+5)I_{4k+4} + (4k+3)I_{4k+2} = \frac{\pi}{2} - \frac{1}{4k+4}$ **B1**

$(4k+3)I_{4k+2} + (4k+1)I_{4k} = \frac{\pi}{2} - \frac{1}{4k+2}$ **B1**

Subtracting:

$(4k+5)I_{4k+4} = (4k+1)I_{4k} + \frac{1}{4k+2} - \frac{1}{4k+4}$ **M1**

$= A - \frac{1}{2}\sum_{r=1}^{2k}(-1)^r \frac{1}{r} + \frac{1}{4k+2} - \frac{1}{4k+4}$ **M1** Use of assumed result

$= A - \frac{1}{2}\sum_{r=1}^{2k}(-1)^r \frac{1}{r} - \frac{1}{2}(-1)^{2k+1}\frac{1}{2k+1} - \frac{1}{2}(-1)^{2k+2}\frac{1}{2k+2}$

$= A - \frac{1}{2}\sum_{r=1}^{2(k+1)}(-1)^r \frac{1}{r}$ **A1** A clear demonstration of how the two extra terms fit must be given **6**