STEP II, 2017, Q13 MS

The first step of part (i) is to find an expression for the probability of getting the correct key on the k^{th} attempt. This can be done from a tree diagram or by using the geometric distribution. From these probabilities an expression for the expectation can be found which is strongly related to the binomial expansion suggested.

Part (ii) also needs to start with an expression for the probability of getting the correct key on the k^{th} attempt. This can be found by telescoping expressions from a tree diagram or just using the symmetry of the situation: each possible selection is equally likely to find the correct key. An expression can again be found and simplified for the expectation.

Part (iii) : A tree diagram type approach forms a series of telescoping fractions, simplifying to the given expression. Pulling out a factor of $(n-1)$ from the expression for the expectation leaves a series of partial fractions which can be written as the difference between the infinite sum given and a finite sum. The difference between an infinite quantity and a finite quantity must be infinite.

(i)		
$P(\text{correct key on } k^{\text{th}}\text{attempt}) = \frac{1}{n}\left(1 - \frac{1}{n}\right)^{k-1}$	M1A1	M1 for any attempt relating to the geometric distribution – e.g. missing first factor or power slightly wrong.
$= pq^{k-1}$ Where $p = \frac{1}{n}, q = 1 - \frac{1}{n}$		Although not strictly necessary, you may see this substitution frequently
Expected number of attempts is given by $p + 2pq + 3pq^2 \ldots$	M1	May be written in sigma notation
$= p(1 + 2q + 3q^2 \ldots)$		
$= p(1-q)^{-2}$	M1	Linking to binomial expansion
$= \frac{p}{p^2} = \frac{1}{p}$		
$= n$	A1	
	[5]	
(ii)		
$P(\text{correct key on } k^{\text{th}}\text{attempt}) = \frac{1}{n}$ for $k = 1 \ldots n$	B1	
Expected number of attempts is given by $\frac{1}{n} + \frac{2}{n} + \frac{3}{n} \ldots + \frac{n}{n}$	M1	
$= \frac{n+1}{2}$	M1A1	M1 for clearly recognising sum of integers / arithmetic series.
	[4]	
(iii)		
$P(\text{correct key on } k^{\text{th}}\text{attempt})$ $= \frac{n-1}{n} \times \frac{n}{n+1} \times \frac{n+1}{n+2} \ldots \times \frac{1}{n+k-1}$	M1 A1	M1 for an attempt at this, possibly by pattern spotting the first few cases. Condone absence of checking $k = 1$ case explicitly.
$= \frac{n-1}{(n+k-2)(n+k-1)}$	M1 AG	M1 for attempting telescoping (may be written as an induction)
$= (n-1)\left(\frac{-1}{n+k-1} + \frac{1}{n+k-2}\right)$	M2 A1	Attempting partial fractions (This may be seen later)
	[6]	
Expected number of attempts is given by $(n-1)\sum_{1}^{\infty}\left(\frac{k}{n+k-2} - \frac{k}{n+k-1}\right)$	M1	
$= (n-1)\left[\left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{2}{n} - \frac{2}{n+1}\right) + \left(\frac{3}{n+1} - \frac{3}{n+2}\right) \ldots\right]$		
$= (n-1)\left[\frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1} \ldots\right]$	M1A1	M1 for attempting telescoping
$= (n-1)\left(\sum_{r=1}^{\infty}\frac{1}{r} - \sum_{r=1}^{n-2}\frac{1}{r}\right)$	B1	
In the brackets there is an infinite sum minus a finite sum, so the result is infinite.	E1	
	[5]	