STEP II, 2017, Q13

13 In a television game show, a contestant has to open a door using a key. The contestant is given a bag containing n keys, where $n \geqslant 2$. Only one key in the bag will open the door. There are three versions of the game. In each version, the contestant starts by choosing a key at random from the bag.

(i) In version 1, after each failed attempt at opening the door the key that has been tried is put back into the bag and the contestant again selects a key at random from the bag. By considering the binomial expansion of $(1-q)^{-2}$, or otherwise, find the expected number of attempts required to open the door.

(ii) In version 2, after each failed attempt at opening the door the key that has been tried is put aside and the contestant selects another key at random from the bag. Find the expected number of attempts required to open the door.

(iii) In version 3, after each failed attempt at opening the door the key that has been tried is put back into the bag and another incorrect key is added to the bag. The contestant then selects a key at random from the bag. Show that the probability that the contestant draws the correct key at the kth attempt is

$$\frac{n-1}{(n+k-1)(n+k-2)}\,.$$

Show also, using partial fractions, that the expected number of attempts required to open the door is infinite.

You may use without proof the result that $\displaystyle\sum_{m=1}^{N} \frac{1}{m} \to \infty$ as $N \to \infty$.